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RESUMO/ABSTRACT

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In recent research we can observe that statistical extreme value theory has been successfully used for modeling stock index prices log returns, since there is empirical evidence that all important samples exhibit heavy tail behaviour.
However, the evidence for goodness-of-fit of an extreme value model is thin, and important empirical characteristics such as the V aR or the expected shortfall show that there may exist a aw in the reasoning leading to the preference for the classical long-tailed Gumbel or Fréchet extreme value distributions; this is not a big surprise since the iid hypothesis leading to those models doesn't apply.
On the other hand, the classical normal model has very light tails, which clearly do not provide a good fit to the data. Therefore, the BASEL II recommendations show in general a shift from the normal towards more realistic models, keeping however an inverse square root scale when dealing with the value at risk at horizon h which is a remant of the normal modeling framework.
We prove that scale mixtures of normal distributions, that can arise when dealing with maxima of non identical normal random variables, can indeed have a very heavy tail, and therefore that they may provide much better patterns to model log returns of stock index prices. We present empirical evidence, analyzing the PSI, which are the main basis for financial decisions in the Portuguese market.

Keywords: Financial Series; Value-at-Risk; Mixtures of Normal Random Variables, Basel II.

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Heavy Tails and Mixtures of Normal Random Variables

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Abstract

In recent research we can observe that statistical extreme value theory has been successfully used for modeling stock index prices log returns, since there is empirical evidence that all important samples exhibit heavy tail behaviour.

However, the evidence for goodness-of-fit of an extreme value model is thin, and important empirical characteristics such as the VaR or the expected shortfall show that there may exist a flaw in the reasoning leading to the preference for the classical long-tailed Gumbel or Fréchet extreme value distributions; this is not a big surprise since the iid hypothesis leading to those models doesn't apply.

On the other hand, the classical normal model has very light tails, which clearly do not provide a good fit to the data. Therefore, the BASEL II recommendations show in general a shift from the normal towards more realistic models, keeping however an inverse square root scale when dealing with the value at risk at horizon h which is a remant of the normal modeling framework.

We prove that scale mixtures of normal distributions, that can arise when dealing with maxima of non identical normal random variables, can indeed have a very heavy tail, and therefore that they may provide much better patterns to model log returns of stock index prices. We present empirical evidence, analyzing the PSI, which are the main basis for financial decisions in the Portuguese market.

Keywords: Financial Series; Value-at-Risk; Mixtures of Normal Random Variables, Basel II.

JEL Classifications: C16, G32.

1. Introduction

Probability and Statistics are tools of the utmost importance to extract knowledge from information, and statistical models play an important role in the "experimental method". As in many other branches of Science, the normal or Gaussian model ruled almost unchallenged in Economy and Finance, due to its ubiquity warranted by the central limit theorem, the simplicity of concepts such as volatility, and the robustness of the linear model. The portfolio selection method of Markovitz (1959), Sharpe's (1964) market equilibrium model, and Black and Scholes (1973) option pricing theory aren't but three relevant examples of path breacking developments taking a parent normal model as granted.

This state of the art collapsed with the widespread use of computers, which provided exuberant evidence that skewness and kurtosis of empirical data couldn't support a normal fit in many instances of modeling financial returns.

In risk assessment, new ways of dealing with evidence provided by extreme order statistics are at the basis of more sophisticated methodologies to avoid extreme losses (Embrechts, 2002). BASEL II contains, in this respect, interesting novelties, such as the recommendation to use the VaR (Value at Risk) - in short, a high quantile that we wouldn't like to upcross - and the expected shortfall as sample characteristics that cannot be ignored in risk management. These recommendations are the natural consequence of the general admission that heavy tailed models provide much better fit than the normal model. A sound overview of VaR methodologies is Jorion (2001), cf. also Danielsson et al. (1998) and Gomes and Pestana (2007).

However, BASEL II recommendations maintained some remains of the past normal model ubiquity, namely the use of using $\sqrt{n}$ for scaling, in the computation of the empirical VaR.

Our purpose is two fold:

1. To begin with, we use empirical characteristics (VaR, expected shortfall) to assess the validity of the normal and of the general extreme value model (a parametrized expression due to von Mises (1936), that encompasses the Gumbel, the Weibull and the Fréchet laws); due to the confidentiality of banking data, we use a virtual portfolio, with a composition mirroring the $PSI20$ (Portuguese

Stock Index). Anticipating our findings: neither the normal model (due to symmetry and light tailedness) nor the extreme value models (due to instabilities in parameter estimation) provide useful fit to the data.

2. As none of the models seems adequate, we tried a different approach: we have established that scale mixtures of normal random variables have heavier tails than the normal variable, and henceforth may serve as an intermediate solution to the modeling of peaks over thresholds, and the estimation of unusually high quantiles. This approach, providing a single mixture model to all data, is philosophically nearer to the views of Mandelbrot on modeling finance data, and accounts for long-range dependence, an important issue in economic time series.

Section 2 is an empirical assessment of the normal and extreme values models in risk management, using a virtual portfolio with a composition similar to the $PSI20$, through the use of VaR and expected shortfalls. Section 3 establishes that scale mixtures of normal random variables have heavier tails than the normal, and restates the most striking conclusions.

2. Empirical evidence

As explained in the Introduction, we consider a portfolio whose composition mirrors the $PSI20$. We use the sequence of daily returns from 1995.01.02 until 2005.06.30.

As there is no prior evidence favouring any model, our first approach is a non-parametric analysis; we compute empirical VaR and ACR (amount of capital in risk) that should have been built up quarterly from the 5th quarter onwards. In other words, according to BASEL II recommendations, 1995 data is the starting information, which should be updated quarterly. Each update uses the previous value, to be corrected using the former 246 daily observations. The skewness (0.55) and kurtosis (6.96) of the data seriously impair the use of a Gaussian model (whose skewness and kurtosis are, both, 0). The p-values when testing normal fit using either the Shapiro-Wilks test and the Kolmogorov-Smirnov tests are less than 0.0005, confirming that the normal provides very poor fit.

On the other hand, the sequence of log-returns

$$Y_{t+1} = \frac{L_{t+1}}{V_t} = -\frac{V_{t+1} - V_t}{V_t} = 1 - \frac{V_{t+1}}{V_t} \approx -\ln\left[1 - \left(1 - \frac{V_{t+1}}{V_t}\right)\right] = -\ln\left(\frac{V_{t+1}}{V_t}\right)$$

where V_t denotes the portfolio value at time t, has smoother behaviour. However, the normal fit is, once again, clearly rejected.

The VaR is a high quantile, more precisely

$$VaR_q(\boldsymbol{L}(t+1)) = F^{\leftarrow}_{\boldsymbol{L}(t+1)}(1-q) = \inf\left\{\ell :\ F_{\boldsymbol{L}(t+1)}(\ell) \geq 1-q\right\}$$

We now recall that BASEL II - clearly retaining features from the normal model - gives a rule of thumb to compute the value at risk with horizon h, i.e., the quantile VaR^h_q which has probability q of being exceeded in the next h observations, as

$$VaR^h_q = \sqrt{h}\, VaR^1_q = VaR_q$$

a very inadequate formula if the true parent model has heavy tails. The values generally adopted are $q = 0.01$ and an horizon $h = 10$.

This new risk assessment tool is then used to compute the quarterly amount of capital in risk (ACR) which financial institutions are requested to keep to face eventual extreme losses; ACR is defined in terms of the annual (in practice, 250 observations) VaR at the end of the previous quarter, and the average VaR in the previous quarter (in practice, 60 observations), as the maximum, for the k-th quarter,

$$MCR_k = \max\left\{\sqrt{10}\ VaR_{250+60k-1},\ \frac{\sqrt{10}}{60}\sum_{j=0}^{59} VaR_{250+60k-1-j}\, B_\ell\right\}$$

where the correction factor B_ℓ is defined on the following grounds:

The number of exceedences of the MCR_k is a binomial random variable with parameters $h = 10$ and $q = 0.01$, and therefore the expected number of upcrossings is 0.1, and the standard deviation is $\sigma = \sqrt{10 \times 0.01 \times 0.99} \approx 0.315$.

Hence the number of exceedences is an interesting guideline for the assessment of the goodness of the model. This can be formalized using the concept of expected shortfall:

Let X be a continuous random variable with finite expectation; the expected shortfall at the confidence level $1-q$ is:

$$ES_q = \mathbb{E}\left(X \mid_{X \geq VaR_q}\right)$$

Integrating

$$ES_q = \frac{\int_{VaR_q}^{\infty} x \, \mathrm{d}F_X(x)}{q} = VaR_q + \frac{\int_{VaR_q}^{\infty} [1 - F_X(x)] \, \mathrm{d}x}{q}$$

and henceforth, under a normal model assumption, standardizing, we get

$$ES_q = \frac{\int_{VaR_q}^{\infty} x \, \mathrm{d}F_X(x)}{q} = \mu + \sigma \, \frac{\int_{\Phi^{-1}(1-q)}^{\infty} z \, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{z^2}{2}} \, \mathrm{d}z}{q} = \mu + \sigma \, \frac{\phi\left(\Phi^{-1}(1-q)\right)}{q},$$

where ϕ denotes the standard normal probability density function, and Φ the corresponding cumulative probability distribution function.

Using standard routines, in a retrospective study of the log-return data, under a normal fit we get, for extreme risk levels, an expected number of exceedances E_q of the upper confidence bound L_q for exceedances under the model that clearly underevaluates the observed number of exceedances O_q which in fact occurred:

Table 1: VaR_q, ES_q and exceedances, at several risk levels, with global data and annual moving windows.

	Risk level	5%	4%	3%	2,5%	2%	1%	0.5%
Global Data	VaR_q (x 100 %)	1,73	1,84	1,98	2,06	2,16	2,45	2,72
	ES_q (x 100 %)	2,17	2,27	2,39	2,46	2,55	2,81	3,05
	E_q	131,00	104,80	78,60	65,50	52,40	26,20	13,10
	L_q	152,87	124,46	95,71	81,16	66,45	36,18	20,18
	O_q	113	103	84	76	67	51	40
Annual Moving Windows	E_q	118,75	95,00	71,25	59,38	47,50	23,75	11,88
	L_q	139,57	113,72	87,54	74,29	60,87	33,25	18,61
	O_q	129	112	94	87	82	52	42

The graphical displays bellow plainly shows our point: using the normal fit the credit institution is making exceedingly high provisions to face the risk, and this isn't a clever investment.

This clearly shows that the normal assumption isn't adequate. The longstanding claim that a proper fit ought to have heavy tails leads naturally to alternative

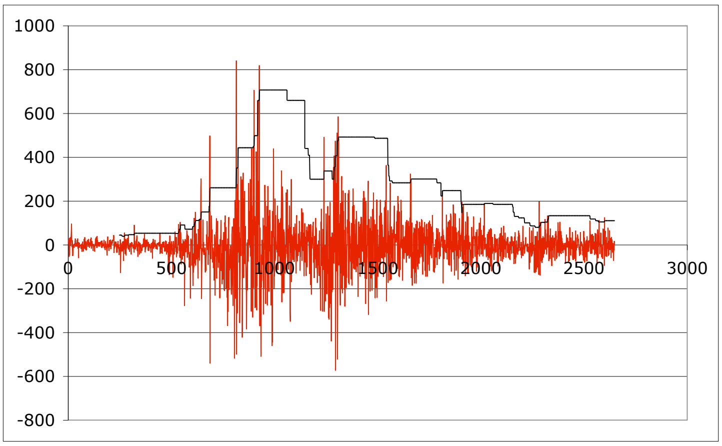

Figure 1: Daily loss (red) and provisions with the $VaR_{0.01}$ (black) computed using the 246 daily observations which have preceded them.

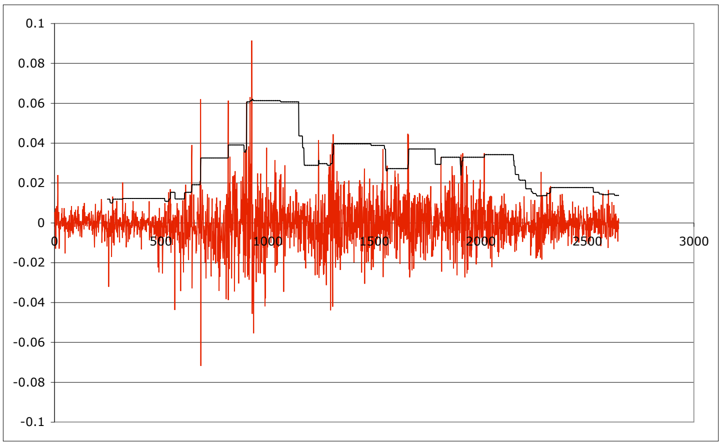

Figure 2: Daily re-exchange (red) in comparison with the correspondent $VaR_{0.01}$ (black), computed using the 246 daily re-exchange which have preceded them.

modeling with extreme value laws, or else with Pareto laws (for each extreme value distribution, we can find a equivalent tail generalized Pareto distribution).

The routines are much more complicated, and exceptional software such as *R* (version 2.4.0, and packages *evir* and *extremes*), and doesn't cope well with parameter estimation for the general extreme value distribution, as extensive simulations we have done confirm. The results of a similar retrospective fitting of the general extreme value distribution (which for long right-tailed data never is of Weibull-type) are in the tables below:

Table 2: Exceedances, Gumbel Model, trimestral adjustment.

Risk level	5%	4%	3%	2,5%	2%	1%	0.5%
E_q	131,00	104,80	78,60	65,50	52,40	26,20	13,10
O_q(*Block of 10 days*)	34	27	21	19	16	14	10
O_q(*Block of 12 days*)	28	24	20	19	19	11	9
O_q(*Block of 15 days*)	24	23	19	18	17	12	7
O_q(*Block of 20 days*)	15	15	11	11	9	8	5

Table 3: Exceedances, Fréchet Model, trimestral adjustment.

Risk level	5%	4%	3%	2,5%	2%	1%	0.5%
E_q	131,00	104,80	78,60	65,50	52,40	26,20	13,10
O_q (*Block of 10 days*)	305	305	305	305	305	305	305
O_q(*Block of 12 days*)	216	216	216	216	216	216	216
O_q(*Block of 15 days*)	211	211	211	211	211	211	211
O_q(*Block of 20 days*)	145	145	145	145	145	145	145

Thus, clearly, modeling with extreme value models is not the correct alternative.

3. **Scale Mixtures of Normal Variables and Conclusions**

Our research confirms that there is empirical evidence against the normal fitting tradition; but, on the other hand, it also shows that models arising in modeling extreme order statistics do not provide the correct answer.

As we have proved that variance mixtures of the normal distribution may have quite heavy tails (Rocha, 2005), those may provide the appropriate frame for modeling log-returns of portfolios. Background information on Probability may be found in Pestana e Velosa (2006); Brilhante et al. (2001) provides information on location/scale inference; and Rocha and Martins (2005) is a good source both for asymptotics and for information on skewness and kurtosis, and their role in approximations.

We sketch the proof, whose main point is to establish that all variance mixtures of the normal law have positive kurtosis, and therefore heavier tails than the normal:

Let Z and Σ be independent random variables, Z symmetrical and $Y > 0$, with moments of order 4. Then the kurtosis of $Z\Sigma$ exceeds the kurtosis of Z.

Proof: From the definition of kurtosis, and since $\mathbb{E}(Z\Sigma) = \mathbb{E}(Z)\,\mathbb{E}(\Sigma) = 0$ (because Z e Σ are independent, and the average of the symmetrical Z, which by hypotheses does exist, must necessarily be 0). With the usual notations for crude and central moments, and for the kurtosis coefficient,

$$3 + \gamma_2(Z\Sigma) = \frac{\mathbb{E}\left[(X\Sigma - \mathbb{E}(X\Sigma))^4\right]}{(\mathrm{var}(X\Sigma))^2} = \frac{\mathbb{E}\left[(X\Sigma)^4\right]}{\left(\mathbb{E}\,(X\Sigma)^2\right)^2} = \frac{\mu_4(X)}{(\mu_2(X))^2}\;\frac{\mu_4'(\Sigma)}{(\mu_2'(\Sigma))^2}$$

From Jensen's inequality, $\left(\mu_4'(\Sigma)\right)^{1/4} \geq \left(\mu_2'(\Sigma)\right)^{1/2}$, one can conclude that $\mu_4'(\Sigma) \geq \left(\mu_2'(\Sigma)\right)^2$, and consequently $3 + \gamma_2(Z\Sigma) \geq \dfrac{\mu_4(X)}{(\mu_2(X))^2}$, meaning, $\gamma_2(Z\Sigma) \geq \gamma_2(Z)$, with equality only if $\Sigma = 1$.

Therefore, any scale mixture of normal random variables has positive kurtosis (unless it is degenerate), and hence its tails are heavier than the normal tails.

This sugests that alternative models should be sought in this huge class. Mandelbrot (1963) recognized the importance of heavy tails, and in Mandelbrot (2002) we may find a collection of papers dealing with the idea that ad hoc changes in models to cope with unexpected changes may be a gross error, disregarding the important Hurst long-range dependence.

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