



Centro de Estudos de
Economia Aplicada do Atlântico

WORKING PAPER SERIES

CEEApIA WP No. 09/2012

Introducing Random Investment in High Speed Rail Transport Valuation

**Gualter Couto
Cláudia Nunes
Pedro Pimentel**

November 2012

Introducing Random Investment in High Speed Rail Transport Valuation

Gualter Couto

Universidade dos Açores (DEG e CEEApIA)

Cláudia Nunes

Instituto Superior Técnico (Departamento de Matemática e CEMAT)

Pedro Pimentel

Universidade dos Açores (DEG e CEEApIA)

Working Paper n.º 09/2012
novembro de 2012

RESUMO/ABSTRACT

Introducing Random Investment in High Speed Rail Transport Valuation

In this paper we extend Couto et al. (2012) high-speed rail (HSR) transport valuation model based on real options analysis, in order to embrace random investment. Optimal timing to invest, value of the option to defer and investment opportunity value are assessed considering uncertainty upon HSR demand and investment expenditures, both following a geometric Brownian motion with jumps driven by Poisson processes. Numerical results are presented, showing the consistence with the former model and the additional uncertainty impact.

JEL classification: D81, D83, D92.

Keywords: options, uncertainty, timing, waiting, investment, high-speed rail.

Gualter Couto
Universidade dos Açores
Departamento de Economia e Gestão
Rua da Mãe de Deus, 58
9501-801 Ponta Delgada

Cláudia Nunes
Instituto Superior Técnico
Departamento de Matemática
Av. Rovisco Pais,
1049-001 Lisboa

Pedro Pimentel
Universidade dos Açores
Departamento de Economia e Gestão
Rua da Mãe de Deus, 58
9501-801 Ponta Delgada

Introducing Random Investment in High Speed Rail Transport Valuation

GUALTER COUTO¹, CLÁUDIA NUNES², PEDRO PIMENTEL¹

¹*Universidade dos Acores, Departamento de Economia e Gestão, and CEEAplA,
R. Mãe de Deus, 9501-801 Ponta Delgada, Portugal,*

²*Instituto Superior Técnico, Departamento de Matemática
Av. Rovisco Pais, 1049-001 Lisboa, Portugal,*

October 31, 2012

Abstract

In this paper we extend Couto et al. (2012) high-speed rail (HSR) transport valuation model based on real options analysis, in order to embrace random investment. Optimal timing to invest, value of the option to defer and investment opportunity value are assessed considering uncertainty upon HSR demand and investment expenditures, both following a geometric Brownian motion with jumps driven by Poisson processes. Numerical results are presented, showing the consistence with the former model and the additional uncertainty impact.

JEL classification: D81, D83, D92.

Keywords: options, uncertainty, timing, waiting, investment, high-speed rail.

1 Introduction

The decision to invest in high-speed rail requires huge financial resources. The financial effort depends upon the technology progress and production costs which influences heavily the investment expenditures. Nowadays the uncertainty regarding the technology development and production costs plays an important role in the decision to invest. In irreversible investment decisions, like HSR, is crucial to be acquainted for the uncertainty impact regarding the investment expenditures.

From the later 70's until today we found a growing literature regarding the importance of new investment valuation framework capable of integrate the uncertainty in optimal investment decisions. An optimal investment decision invariably depends on the option to defer, among others options. Tourinho (1979) was one of the first researchers to show that the possibility of

delaying natural resources reserve's operations (such as crude oil) could be studied and valued as an option. The timing to invest has cognized theoretical developments when McDonald and Siegel (1986) developed a rule to determine the optimal timing to invest when the projects' value and the investment's expenditure are both stochastic. Simultaneously, they have quantified the lost value when the investment is done in a suboptimal time. Despite the significant research about real options analysis focusing in the transports investments (Rose (1998); Brandao (2002); Smit (2003); Salahaldin and Granger (2005); Pereira et al. (2006)), the work of Bowe and Lee (2004) seems pioneer in the analysis of a railway investment. Nevertheless, Bowe and Lee (2004) compute the option's value using numerical solutions provided by binomial analysis. Related to analytical developments aiming closed-form solution's regarding the ROA's application to railways investment, Pimentel et al. (2012) develops a framework to value under uncertainty a HSR investment with demand and investment expenditures following a geometric Brownian motion. Couto et al. (2012) introduced conjecture shocks upon HSR demand on a single uncertainty valuation framework.

Compared to the literature, we will combine two uncertainty, shortcoming on Couto et al. (2012), and conjectural shocks, shortcoming on Pimentel et al. (2012), in a single model. Like in these two previous papers we assume that HSR demand follows a geometric Brownian motion facing jumps of random magnitude. We also assume that investment expenditures follow a geometric Brownian motion with the possibility of jumps, driven by a Poisson process. The Poisson process upon HSR demand captures unanticipated future events, like a HSR line extension from a hub forcing a boost in HSR demand sometime in the future. The Poisson process upon investment expenditures captures unanticipated future events like a technological shock forcing a shift in investment expenditures sometime in the future.

This paper introduces in transport valuation field the HSR investment analysis in continuous time with two stochastic processes facing random shocks, providing some closed-form solutions. Our model was developed based on the project of HSR in Portugal, although it's flexible enough to be applied to other similar investment projects.

The remainder of the paper is organized as follows: Section 2 presents the framework rationale and develops the valuation framework. Section 3 focuses on the optimal policy given the ability to defer the investment. Section 4 focuses on the value of the investment opportunity. Section 5 provides the corresponding numerical results and the parallel economic rationale. Finally, in Section 6 we present the main conclusions and recommendations for future extensions.

2 The model

The investment in a HSR can be seen as an optimal stopping problem, where stopping means that the investment decision is implemented. Thus, the main flexibility for the project's owner comes from holding an option to defer. Following McDonald and Siegel (1986); Pimentel et al. (2012); Couto et al. (2012), the optimal timing to invest rule is derived in this paper.

Although the main source of uncertainty is related to HSR demand level subject to abrupt

changes caused by unexpected conjecture shocks, as in Couto et al. (2012), uncertainty upon investment expenditures play also an important role in the optimal investment decision. If future improvements in technology and production efficiency tend to change investment expenditures, we expect a certain influence upon the optimal timing to implement de investment, compared to lack of technology and production improvement expectation. Of course future technology and production improvements are uncertainty. So this uncertainty must affect to some extent the value of the option do defer and the optimal timing to invest.

In this section we introduce the mathematical model, along with the assumptions that we use in order to derive the optimal investment policy. For that, we start by presenting the dynamics of the stochastic processes involved, namely, the demand and the investment expenditures processes. Then we also present the valuation model, where we specify the costs and all the assumptions associated with this decision problem.

2.1 Dynamics of the processes involved

We assume the following dynamics for the two sources of uncertainty:

- The demand process, hereby denoted by $X = \{X_t, t \geq 0\}$

$$\frac{dX_t}{X_t} = \mu_X + \sigma_X dW_t^X + U_t^X dN_t^X \quad (1)$$

where X_t is the annual demand at year t , $W^X = \{W_t^X, t \in \mathfrak{R}^+\}$ is a Brownian motion, $N^X = \{N_t^X, t \in \mathfrak{R}^+\}$ is a Poisson process with rate λ_X , independent of $\{X_t, t \in \mathfrak{R}^+\}$, and $\{U_t^X, t \in \mathfrak{R}^+\}$ is a family of independent and identically random variables, whose values are also independent of X , N^X and W^X .

- The investment expenditures, denoted by $I = \{I_t, t \geq 0\}$

$$\frac{dI_t}{I_t} = \mu_I + \sigma_I dW_t^I + U_t^I dN_t^I \quad (2)$$

where I_t is the investment expenditures at year t , $W^I = \{W_t^I, t \in \mathfrak{R}^+\}$ is a Brownian motion, $N^I = \{N_t^I, t \in \mathfrak{R}^+\}$ is a Poisson process with rate λ_I , independent of $\{I_t, t \in \mathfrak{R}^+\}$, and $\{U_t^I, t \in \mathfrak{R}^+\}$ is a family of independent and identically random variables, whose values are also independent of I , N^I and W^I .

If $\sigma_{W_t^X, W_t^I} = \sigma_{X,I}, \forall t$ then it means that we assume that the correlation of the Brownian motion underlying the dynamics of the demand process and the Brownian motion underlying the dynamics of the investment process is constant in time. Moreover, if $\sigma_{X,I} = 0$ then we are assuming that these processes are uncorrelated and thus, taking into account the normality of the Brownian motion, they are independent. From the moment that the decision maker decides to invest in the HSR, the evolution of both X and I become irrelevant.

We note that Equations (1) and (2) reflect that both the demand process X and the investment expenditures process I follow a geometric brownian motion but that at some time instants, driven by (independent) Poisson processes, there are jumps (in the demand level and/or in the investment expenditures), and these jumps can be deterministic or random, according to the assumptions present in the model, and that will be hereafter analysed.

In Equation (1) the parameters have the following meaning: μ_X represents the growth rate of the demand level, and σ_X represents the respective volatility. We assume that both parameters are constant in time. The same interpretation and assumptions hold for the investment expenditures dynamics, with the obvious changes.

Following Merton (1976a), the solution of Equation (1) is as follows:

$$X_t = x_0 e^{(\mu_X - \frac{\sigma_X^2}{2})t + \sigma_X W_t^X} \left(\prod_{j=1}^{N_t^X} (1 + U_j^X) \right) \quad (3)$$

whereas the solution of Equation (2) is:

$$I_t = i_0 e^{(\mu_I - \frac{\sigma_I^2}{2})t + \sigma_I W_t^I} \left(\prod_{j=1}^{N_t^I} (1 + U_j^I) \right) \quad (4)$$

where x_0 and i_0 are, respectively, the present HSR demand and the present investment expenditures (i.e., the investment needed to undertake the construction of the HSR, if accomplished today). We note, however, that the results expressed in Equations (3) and (4) hold if, and only if, the Brownian motions W^X and W^I are independent.

In case the involved Brownian motions, W^X and W^I , are dependent, then one has to solve a system of two stochastic differential equations. In the absence of jumps in the demand and investment processes, it is still possible to find a closed expression for the solution of (1) and (2), given by:

$$X_t = x_0 e^{(\mu_X - \frac{\sigma_X^2}{2})t + \sigma_X \sum_{j=1}^2 h_{1j} Z_{j,t}} \quad (5)$$

$$I_t = i_0 e^{(\mu_I - \frac{\sigma_I^2}{2})t + \sigma_I \sum_{j=1}^2 h_{2j} Z_{j,t}} \quad (6)$$

where $Z_t = (Z_{j,t})$ is such that $\{Z_{1,t}\}$ and $\{Z_{2,t}\}$ are independent Brownian motions related with W^X and W^I as follows:

$$W_t = (W_t^X, W_t^I) = H(Z_{1,t}, Z_{2,t})$$

Here $H = (h_{ij})$, with $h_{11} = h_{22} = 1$ and $h_{12} = h_{21} = \sigma_{XI}$; see Kloeden and Platen (1999).

In the presence of jumps, one cannot find a closed expression for the solution of the system of the two stochastic differential equations. Therefore, from now on, we assume that $\sigma_{X,I} = 0$. Therefore, in that case we assume that the processes of demand and investment expenditures are independent.

In the rest of the paper we need some statistical moments of the demand level and the investment; in fact we need to derive the following expected value of a power p of X_t : $E[X_t^p]$, where

for now p is a real number (latter on it will have some explicit meaning). Using the fact that the conjuncture shocks are exogenous (and therefore W_t and N_t are independent random variables), we can derive the following expression:

$$\begin{aligned} E[X_t^\theta] &= x_0^\theta e^{\theta(\mu_X - \frac{\sigma_X^2}{2})t} E[e^{\theta\sigma_X W_t^X} \prod_{j=1}^{N_t^X} (1 + U_j^X)^\theta] \\ &= x_0^\theta e^{\left[\theta\mu_X + \frac{\sigma_X^2}{2}\theta(\theta-1) + \lambda_X(E[(1+U^X)^\theta] - 1)\right]t} \end{aligned} \quad (7)$$

We note that Equation (7) follows from the moment generator function of a normal distribution, the moment generator function of a Poisson distribution and the Wald equation, Ross (1996). Finally, as we stated before, $\{U_t^X\}$ is a family of random variables independent and identically distributed, and we denote its common distribution by U^X . Then it means that every time there is a jump in the demand level, it can take any value according to the distribution of U^X .

Similarly, the following result holds for the investment expenditures:

$$E[I_t] = i_0 e^{[\mu_I + \lambda_I(E[(1+U^I)] - 1)]t} \quad (8)$$

2.2 The valuation model

The goal of the decision-maker is to choose optimally the investment time, in order to maximize the expected value of the investment in the HSR. Let τ denote such time. Following Couto et al. (2012), the expected value of the investment is given by:

$$v(\tau) = \sup_T E \left[\int_0^{T+n} X_t V_{\text{before}}(X_t) e^{-\rho t} dt + \int_{T+n}^{\infty} X_t V_{\text{after}}(X_t) e^{-\rho t} dt \middle| X_0 = x_0, I_0 = i_0 \right] \quad (9)$$

where by V_{before} (V_{after}) we denote the value that each user associates to a railway trip before (after) the implementation of the investment, which we assume to be given by:

$$V_{\text{before}}(X_t) = m_t - \beta_0 X_t^\alpha - a X_t^\alpha \quad (10)$$

$$V_{\text{after}}(X_t) = m_t - \beta_1 X_t^\alpha - \frac{\varphi_t}{X_t} - \frac{\rho I_t e^{\rho n}}{X_t} \quad (11)$$

where n , in Equation (11), is the number of years that the HSR will start operating, after the investment decision (time to build effect).

In Equations (10, 11), we use the following notation and assumptions:

- m_t is the individual disposable income at time t ;
- $\beta_0 X_t^\alpha$ ($\beta_1 X_t^\alpha$) represents the value of travel value for before (after) the implementation if the HSR, with $\beta_0 \geq \beta_1$. The difference between β_0 and β_1 reflects the decrease in travel time due to the implementation of the HSR.

- aX_t^α is the value of the travel fare. The current valuation framework implicitly assumes that each user will bear his/her part of the investment expenditure. Hence, along with the benefit from decrease in travel time, Equation (11) considers the saved conventional travel fare. A fair HSR travel fare is already implicitly considered in the valuation framework.
- $\frac{\varphi_t}{X_t}$ is the operating cost per unit of time at time t for each HSR user, and $\frac{\rho I_t}{X_t}$ represents the investment expenditure per unit of time for each HSR user. Following Pimentel et al. (2012), we assume that the operating cost is in fact a proportion (l , say) of the total investment expenditures, so that $\varphi_t = lI_t$.

Remark that we assume an iso-elastic utility function, both for the value of travel value and for the travel fare. Moreover, α represents the elasticity between the fare value and the HSR demand, and the elasticity between the total value of travel time and the HSR demand, X_t . So we assume that both elasticities are equal, meaning that the value of the travel time for the user and the value of the travel fare increase at the same rate. This can certainly be assumed in stable economies, for instance.

Plugging Equations (10,11) in Equation (9), we re-write the value function, v , that depends on the time that the HSR is implemented, T , is as follows:

$$v(T) = E_{(x_0, i_0)} \left[\int_0^\infty (m_t X_t - \beta_0 X_t^{\alpha+1} - a X_t^{\alpha+1}) e^{-\rho t} dt + \int_{T+n}^\infty ((\beta_0 - \beta_1 + a) X_t^{\alpha+1} - (l + \rho e^{\rho n}) I_t) e^{-\rho t} dt \right] \quad (12)$$

where $E_{(x_0, i_0)}[\dots]$ denotes the expected value of the expression inside the brackets, conditional on $X_0 = x_0$ and $I_0 = i_0$. Therefore, as the first member of Equation (12) does not depend on T , the maximization problem that we need to solve is the following:

$$v(\tau) = \sup_T v(T) = \sup_T E_{(X_T, I_T)} \left[\int_{T+n}^\infty ((\beta_0 - \beta_1 + a) X_t^{\alpha+1} - (l + \rho e^{\rho n}) I_t) e^{-\rho t} dt \right] = \sup_T \int_T^\infty ((\beta_0 - \beta_1 + a) E_{X_T}[X_{t+n}^{\alpha+1}] - (l + \rho e^{\rho n}) E_{I_T}[I_{t+n}]) e^{-\rho t} dt \quad (13)$$

where in Equation (13) we use the argument that T is a stopping time for $\{(X_t, I_t)\}$, and thus one may use the strong Markov property (Oksendal (1998)). Moreover, $E_{(X_T, I_T)}[\dots]$ denotes the expected value operator, conditional on X_T and I_T . Furthermore, we also assume independence of the processes X and I , and that is the reason why we compute $E_{X_T}[X_t^{\alpha+1}]$ and $E_{I_T}[I_t]$.

3 Optimal policy

In this section we derive the optimal investment policy for this case. Recall that τ denotes the (random) time when it is optimal to invest in the HSR. Following the usual approach of Real

Options (see, for example, Dixit and Pindyck (1994), Huisman (2000)) we use, instead the level of demand and investment where the investment in the HSR is an optimal decision. Therefore we denote by (x^*, i^*) such levels, and we call them the trigger values.

Thus, regarding the optimal value function $v(\tau)$, we use instead the function $u(x^*, i^*)$, such that:

$$u(x^*, i^*) = \sup_{x,i} \int_0^\infty ((\beta_0 - \beta_1 + a)E_x[X_{t+n}^{\alpha+1}])e^{-\rho(t+n)} - (le^{-\rho n} + \rho)E_i[I_{t+n}]e^{-\rho t} dt \quad (14)$$

Using Equations (7,8), we can compute the value function u as follows:

$$\begin{aligned} u(x^*, i^*) &= \int_0^\infty \left((\beta_0 - \beta_1 + a)(x^*)^{\alpha+1} e^{\left[(\alpha+1)\mu_X + \frac{\sigma_X^2}{2}(\alpha+1)(\alpha+2) + \lambda_X(E[(1+U^X)^{\alpha+1}] - 1) \right] (t+n)} \right) e^{-\rho(t+n)} \\ &\quad - (le^{-\rho n} + \rho)i^* e^{\left[\mu_I + \lambda_I(E[(1+U^I)] - 1) \right] (t+n)} \right) e^{-\rho t} dt \\ &= A(x^*)^\theta - Bi^* \end{aligned} \quad (15)$$

with $\theta = \alpha + 1$, where

$$A = \frac{(\beta_0 - \beta_1 + a)e^{((\theta)\mu_X + \frac{\sigma_X^2}{2}(\theta-1)\theta + \lambda_X E[(1+U_X)^\theta] - \lambda_X - \rho)n}}{\rho - \left(\theta\mu_X + \frac{\sigma_X^2}{2}(\theta-1)\theta + \lambda_X(E[(1+U_X)^\theta] - 1) \right)} \propto (\beta_1 - \beta_0 + a) \quad (16)$$

$$B = \frac{(le^{-\rho n} + \rho) e^{(\mu_I + \lambda_I(E[(1+U^I)] - 1))n}}{\rho - \mu_I - \lambda_I(E[(1+U^I)] - 1)} \quad (17)$$

Therefore, the total (discounted) benefits that results from the investment decision in the HSR are given by $u(x, i)$, for $x > x^*$ and $i > i^*$, the so-called *stopping region*.

On the other side, in the so-called *continuation region* (where investment is not optimal) the Bellman equation (Dixit and Pindyck (1994)) turns out to be given by:

$$\rho u(X_t, I_t) dt = E[du(X_t, I_t)] \quad (18)$$

meaning that in a time interval dt the total return that comes from the investment implementation is equal to the expected investment capital gains.

Therefore one needs to compute the (stochastic) differential form of u , taking now into account that u depends on two random variables (X_t and I_t) and that both have jumps (so that their sample path is not continuous with probability one). Thus we cannot apply Itô's formula straight head, but, instead, we use Merton (1976b) result along with Bjork (2009) (Theorem 4.16), so that we end up with the following differential form for u :

$$\begin{aligned} du(X, I) &= \frac{\partial u(X, I)}{\partial X} (dX) + \frac{\partial u(X, I)}{\partial I} (dI) + \\ &\quad \frac{1}{2} \left[\frac{\partial^2 u(X, I)}{\partial X^2} (dX)^2 + \frac{\partial^2 u(X, I)}{\partial I^2} (dI)^2 + 2 \frac{\partial^2 u(X, I)}{\partial X \partial I} (dX)(dI) \right] \\ &\quad + \lambda_X E[u((1+U_X)X, I)] + \lambda_I E[u(X, (1+U_I)I)] \end{aligned} \quad (19)$$

Now we plug in Equation (19) dX and dI given by Equations (1-2), we use the fact:

$$\begin{aligned}(dX)^2 &= \sigma_X^2 X^2 dt \\ (dI)^2 &= \sigma_I^2 I^2 dt \\ (dX)(dI) &= \sigma_X \sigma_I IX Cov(W^X, W^I) dt = \sigma_X \sigma_I \sigma_{X,I} IX dt = 0\end{aligned}$$

where $\sigma_{X,I} = Cov(W^X, W^I) = 0$ (recall that in the presence of the jump process either in X or in I , we have to assume the independence of the Brownian motions), and $E[dW^X] = W[dW^I] = 0$. Finally we plug Equation (19) in Equation (18), and as a result we end up with the following stochastic differential equation:

$$\begin{aligned}ru(X, I) - \left(X \frac{\partial u(X, I)}{\partial X} \mu_X + I \frac{\partial u(X, I)}{\partial I} \mu_I \right) - \frac{1}{2} \left(X^2 \frac{\partial^2 u(X, I)}{\partial X^2} \sigma_X^2 + I^2 \frac{\partial^2 u(X, I)}{\partial I^2} \sigma_I^2 \right) \\ + \lambda_X E[u((1 + U_X)X, I)] + \lambda_I E[u(X, (1 + U_I)I)] = 0\end{aligned}\quad (20)$$

with has to hold for every X and I with probability 1. Therefore, in order to find the function value u , we need to solve the following PDE:

$$\begin{aligned}ru(x, i) - \left(x \frac{\partial u(x, i)}{\partial x} \mu_X + i \frac{\partial u(x, i)}{\partial i} \mu_I \right) - \frac{1}{2} \left(x^2 \frac{\partial^2 u(x, i)}{\partial x^2} \sigma_X^2 + i^2 \frac{\partial^2 u(x, i)}{\partial i^2} \sigma_I^2 \right) \\ + \lambda_X E[u((1 + U_X)x, i)] + \lambda_I E[u(x, (1 + U_I)i)] = 0\end{aligned}\quad (21)$$

where in Equation (21), the term $E[u((1 + U_X)x, i)]$ regards the expectation with respect to the jump size in the demand process, U_X , whereas the term $E[u(x, (1 + U_I)i)]$ regards the expectation with respect to the jump size in the investment expenditures, U_I .

We admit the following boundary conditions:

$$u(0, i) = 0, \forall i \quad (\text{Initial condition}) \quad (22)$$

$$u(x^*, i^*) = A(x^*)^\theta - Bi^* \quad (\text{Value-matching condition}) \quad (23)$$

$$\frac{\partial u(x, i)}{\partial x} \Big|_{x=x^*, i=i^*} = A\theta(x^*)^{\theta-1} - B \quad (\text{Smooth-pasting condition}) \quad (24)$$

The problem is that it is very hard to solve this equation analytically, and in general one can only use numerical methods. Fortunately is possible to search for a decision criteria relying upon the demand level per unit of investment expenditures, reducing the problem to a single dimension. Therefore, like in Dixit and Pindyck (1994), we propose a change in the variables in such a way that we end up with just one variable. This single variable q represents the ratio between HSR demand and the investment expenditures, as we show next.

So assume that

$$u(x, i) = if \left(\frac{x^\theta}{i} \right) = if(q) \quad (25)$$

where $q = \left(\frac{x^\theta}{i} \right)$ and f is a function to be determined. Successive derivation of the value function

u proposed in Equation (25) gives the following:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \theta x^{\theta-1} f'(q), \\ \frac{\partial u}{\partial i} &= f(q) - \frac{x^\theta}{i} f'(q) = f(q) - q f'(q) \\ \frac{\partial^2 u}{\partial x^2} &= \theta^2 \frac{x^{2\theta-2}}{i} f''(q) + \theta(\theta-1) x^{\theta-2} f'(q) \\ \frac{\partial^2 u}{\partial i^2} &= \frac{x^{2\theta}}{i^3} f''(q) = \frac{q^2}{i} f''(q)\end{aligned}$$

Replacing these expressions in Equation (20) and simplifying, we arrive to the following second-order ordinary differential equation:

$$\begin{aligned}& \frac{1}{2}[\sigma_X^2 \theta^2 + \sigma_I^2] q^2 f''(q) + [\mu_X \theta + \frac{1}{2} \sigma_X^2 \theta(\theta-1) - \mu_I] q f'(q) \\ & - (\rho - \mu_I) f(q) + \lambda_X E\left[\frac{u((1+U_X)x, i)}{i}\right] + \lambda_I E\left[\frac{u(x, (1+U_I)i)}{i}\right] \\ & = \frac{1}{2}[\sigma_X^2 \theta^2 + \sigma_I^2] q^2 f''(q) + [\mu_X \theta + \frac{1}{2} \sigma_X^2 \theta(\theta-1) - \mu_I] q f'(q) \\ & - (\rho - \mu_I) f(q) + \lambda_X E\left[f\left(\frac{(1+U_X)^\theta x^\theta}{i}\right)\right] + \lambda_I E\left[(1+U_I) f\left(\frac{x^\theta}{(1+U_I)i}\right)\right] \\ & = \frac{1}{2}[\sigma_X^2 \theta^2 + \sigma_I^2] q^2 f''(q) + [\mu_X \theta + \frac{1}{2} \sigma_X^2 \theta(\theta-1) - \mu_I] q f'(q) \\ & - (\rho - \mu_I) f(q) + \lambda_X E[f((1+U_X)^\theta q)] + \lambda_I E[(1+U_I) f\left(\frac{q}{(1+U_I)}\right)]\end{aligned}\quad (26)$$

Now, in order to be able to solve Equation (26), we still need to address the terms $E[f((1+U_X)^\theta q)]$ and $E[(1+U_I) f(\frac{q}{(1+U_I)})]$. For that, assume, for the time being, that $\lambda_X = \lambda_Y = 0$, so that in this particular case we look for the solution of the following ODE:

$$\frac{1}{2}(\sigma_X^2 \theta^2 + \sigma_I^2) q^2 f''(q) + (\mu_X \theta + \frac{1}{2} \sigma_X^2 \theta(\theta-1) - \mu_I) q f'(q) - (\rho - \mu_I) f(q) = 0 \quad (27)$$

This problem has already been addressed by Pimentel et al. (2012), and for this equation, assuming that

$$f(0) = 0$$

the solution can be written as the following:

$$f(q) = b q^r \quad (28)$$

where b and r are derived so that boundary conditions (also provided in Pimentel et al. (2012)) are satisfied. For our purposes, we just need to know the general form of the function f . So,

assuming that indeed the solution of Equation (27) is of the form given in Equation (28), we now assume the following:

$$E[f((1 + U_X)^\theta q)] = f(q)E[(1 + U_X)^{\delta_1}] \quad (29)$$

$$E[f((1 + U_I)^{-1} q)] = f(q)E\left[\frac{1}{(1 + U_I)^{\delta_2}}\right] \quad (30)$$

for some value of $\delta_1, \delta_2 \in \mathfrak{R}$. Therefore, the function f is such that it is a solution of the following ODE:

$$\begin{aligned} & \frac{1}{2}[\sigma_X^2 \theta^2 + \sigma_I^2]q^2 f''(q) + [\mu_X \theta + \frac{1}{2}\sigma_X^2 \theta(\theta - 1) - \mu_I]q f'(q) \\ & - (\rho - \mu_I + \lambda_X E[(1 + U_X)^{\delta_1}] + \lambda_I E[\frac{1}{(1 + U_I)^{\delta_2}}])f(q) \end{aligned} \quad (31)$$

We note that Equation (31) is a Cauchy-Euler equation, whose solution is known to be given by:

$$f(q) = a_1 q^{r_1} + a_2 q^{r_2} \quad (32)$$

such that the following boundary conditions are met:

$$f(0) = 0 \quad (33)$$

$$f(q^*) = Aq^* - B \quad (34)$$

$$f'(q^*) = A \quad (35)$$

where Equations (34,35) hold in view of Equations (15) and (25), and r_1 and r_2 are, respectively, the positive and negative solution of the following equation:

$$\frac{1}{2}[\sigma_X^2 \theta^2 + \sigma_I^2]x(x - 1) + [\mu_X \theta + \frac{1}{2}\sigma_X^2 \theta(\theta - 1) - \mu_I]x - \quad (36)$$

$$(\rho - \mu_I + \lambda_X E[(1 + U_X)^{\delta_1}] + \lambda_I E[\frac{1}{(1 + U_I)^{\delta_2}}]) = 0 \quad (37)$$

We remark that as $r_2 < 0$, then $q^{r_2} \rightarrow \infty$ when $q \rightarrow 0$. From the condition $f(0) = 0$, it follows then that $a_2 = 0$. Moreover, using the value-matching condition, we conclude that:

$$a_1 = (Aq^* - B)(q^*)^{-r_1} \quad (38)$$

Therefore, in view of the last developments, it follows that the solution to Equation (31) is:

$$f(q) = [(A(q^*) - B)(q^*)^{-r_1}] q^{r_1} \quad (39)$$

Thus the value q^* such that $f(\cdot)$ is maximum, is the following:

$$q^* = \frac{B}{A} \frac{r_1}{r_1 - 1} \quad (40)$$

We note that this trigger value, q^* , is the optimal ratio between HSR demand (given by x^*) and the investment expenditures.

Therefore the optimal investment policy is the following: for each time t , $I_t = i_t$ and $X_t = x_t$ are known; then compute

$$q_t = \frac{x_t^\theta}{i_t} \quad (41)$$

If $q_t < q^*$, postpone the investment decision; otherwise, invest.

Note that according to the previous rule, once $I_t = i$ is known, the demand level that triggers the investment decision (that, for ease of notation, we denote by x^*) is:

$$x^* = \left(\frac{B}{A} \frac{r_1}{r_1 - 1} i \right)^{1/\theta} \quad (42)$$

For a known investment expenditures at time t , the optimal HSR investment implementation occurs if HSR demand trigger level x^* is reached.

3.1 Investment opportunity value

Considering the investment's value function solution given by Equation (40), for a certain pair of value of the demand and the investment at time $t = 0$, (x_0 and i_0), the investment opportunity value, when $q < q^*$, is given by:

$$f(q) = \begin{cases} \left(\frac{q}{q^*} \right)^{r_1} (Aq^* - B) & q < q^* \\ Aq - B & q \geq q^* \end{cases} \quad (43)$$

In accordance to previous studies (see McDonald and Siegel (1986); Dixit and Pindyck (1994)), from the moment in which the HSR trigger value is reached, q^* , the value of the option to defer is zero. As a result, it is always better to invest and receive in exchange the net present value (NPV).

As long as the optimal timing to invest has not been reached, there is always an inherent value of waiting for new information about the HSR demand and investment expenditures. In this case, the value of the option to defer is given by the difference between the investment opportunity value, $u(x, i)$, and the NPV calculated, using the HSR demand and the investment expenditures at that moment.

4 Numerical Illustration

In this section we present numerical values that illustrate the influence of combining uncertainty upon investment expenditures and demand shocks in the decision about the investment in HSR. The parameter values that we present, as shown in Table (1), are supported by the Portuguese

Government data released on the HSR investment project. We note, however, that in this present paper this numerical illustration is also used as a case study, and therefore could be used in other situations, with another set of parameters values.

Table 1: Base-case parameters for the project

Parameter	Numerical value
x (present demand)	3M
i_0 (present value of investment expenditures)	5,000 M ¹ €
β_0 (value of travel time in conventional railway)	30 €
β_1 (value of travel time in HSR)	10 €
ρ (discount rate)	0.09
μ_X (expected growth rate of the demand process)	0.035
σ_X (standard deviation of the demand process)	0.15
λ_X (jump intensity of the demand process)	0.10
$E[U_X]$ (expected value of the jump size of the demand process)	0.10
μ_I (expected growth rate of the investment expenditures)	0.01
σ_I (standard deviation of the investment expenditures)	0.10
λ_I (jump intensity of the investment expenditures)	0.10
$E[U_I]$ (expected value of the jump size of the investment expenditures)	0.1
n (number of years for the construction)	5
l (fixed operating costs/investment expenditures)	0.018

All the assumptions are similar to Couto et al. (2012) numerical illustration, and therefore supported in the same way. We face 5 billion Euros investment expenditures, 5 years construction period, for a HSR line which intends to reduce the travel time from around 3 hours to around 1 hour, compared to the conventional railway service operating in the same link. Also the value of travel time per hour is set to 20 euros.

Concerning the jumps, we assume different situations, in order to access the impact of the additional uncertainty upon investment expenditures and the jumps and its distribution in the optimal investment policy. Moreover, the parameter λ_X captures unanticipated future events which affects the HSR demand, like a HSR line extension from a hub forcing a boost U_X in HSR demand sometime in the future, whereas λ_I regards unanticipated future events which affects the investment expenditures, like a technological shock forcing a shift U_I in investment expenditures sometime in the future.

Finally, as usual in capital budgeting, the operational fixed cost is set to be a proportion from the investment expenditures.

In the following tables we present the numeric values for this particular illustration. We compare the following situations:

- i) *Deterministic Investment*: in this case we assume that $I_t = i_0 = 5000$. We note that this

corresponds to the case presented in Couto et al. (2012). We assume that

- i.1) No jumps (i.e., the demand process is a continuous process; we denote this situation in the table (2) by nJ);
- i.2) Deterministic jumps (i.e, jumps of magnitude equal to $E[U_X] = 0.1$) occur in the demand level according to a Poisson process (J, in table (2));
- ii) *Stochastic Investment*: in this case $\{I_t\}$ follows the Geometric Brownian motion with jumps driven by a Poisson process. We consider the following cases
 - ii.1) No jumps in either the processes (nJ+nJ, in the table)
 - ii.2) Deterministic jumps (magnitude equal to $E[U_X]$) in the demand process, no jumps in the investment (J+nJ);
 - ii.3) Deterministic jumps in both processes (magnitudes equal to $E[U_X]$ and $E[U_I]$) (J+J)

Remark that for the numerical illustration we choose to assume that, in case there are jumps, their magnitude is deterministic. Moreover, when we assume that there are no jumps (either in the demand or in the investment expenditures), we use the model and equations (namely Equation (42), with A and B given by Equations (16,17) previously derived, setting $\lambda_X = 0$ (in the absence of jumps in the demand) and/or $\lambda_I = 0$ (for the investment expenditures case).

For reasons of space, we consider situation i) presented in Table (2), and situation ii) presented in Table (3). For all cases, we present the HSR demand trigger value, x^* , the investment opportunity value, $v(\cdot)$, the NPV and the value option to defer.

Table 2: HSR investment valuation results, with stochastic demand

	nJ	J
x^* (HSR demand trigger value, in Million)	9.85	9.18
$v(x^*)$ (Invest. Opport. Value, in M €)	1,769.18	6,655.63
NPV (in M €)	-1,620.72	3,568.63
Value of the Option to Defer (in M€)	3,389.88	2,997.99

Table 3: HSR investment valuation results, with stochastic demand and investment expenditures

	nJ+nJ	J+nJ	J+J
x^* (HSR demand trigger value, in Million)	10.82	4.68	4.94
$v(x^*)$ (Invest. Opport. Value, in M €)	1,645.53	987.03	545.86
NPV (in M €)	-2,650.60	-770.51	-2,113.70
Value of the Option to Defer (in M€)	4,296.13	1,757.54	2,659.56

In Table (2) and Table (3) we present the different values obtained regarding the level of demand that justifies the investment implementation (x^*), as well as the investment opportunity value, the NPV and the value of the option to defer. All output values were generated using the same extended model.

With negative NPV the project should not be implemented at the current time, concerning the uncertainty regarding the number of passengers of the new service and the investment expenditures (see Table (3), where the NPV is negative for the 3 cases). Maintaining 'alive' this investment opportunity has a value that ranges from 545.86 M €(for the J+J case) until 1,645.53 M€(for the nJ+nJ case). Compared with a scenario facing uncertainty only in HSR demand, the additional uncertainty upon investment expenditures facing jumps results in lower values for investment opportunity value and NPV.

With uncertainty upon HSR demand and investment expenditures, both facing positive jumps, the optimal decision to invest should only be taken when HSR demand reaches 4.94 million passengers, compared to the 9.19 million passengers without uncertainty upon investment expenditures.

According to the existing literature, in a non-jump scenario with uncertainty upon HSR demand, the additional uncertainty upon investment expenditures (also without jumps) postpone the optimal timing to invest, resulting in a higher value of the option to defer. Nevertheless, in a scenario with two uncertainties, adding jumps (upon HSR demand and/or investment expenditures) result in a significant reduction of the trigger value. Furthermore, we note that the valuation model is very sensitive to the existence/non-existence of jumps, no matter if they occur in the HSR demand or/and in the investment expenditures.

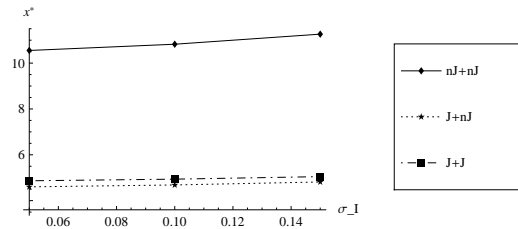


Figure 1: Behavior of x^* as a function of the volatility of the investment expenditures volatility (σ_I)

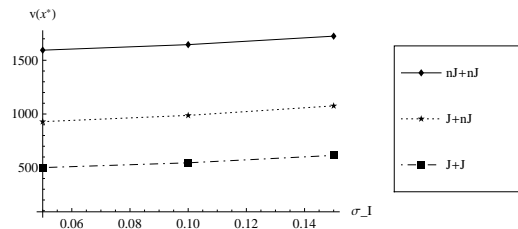


Figure 2: Behavior of the investment opportunity value as a function of the volatility of the investment expenditures volatility (σ_I)

Figures 1 and 2 show the impact on the investment expenditures volatility in the trigger value and in the investment opportunity value. Note that both the trigger and the investment

opportunity value for each case show a near flat behavior for different levels of uncertainty upon investment expenditures. Furthermore, for all the cases the absolute difference of these values is roughly constant with σ_I . Therefore, one may conclude from these numerical values that the valuation model is not very sensitive with respect to the change in the investment expenditures volatility.

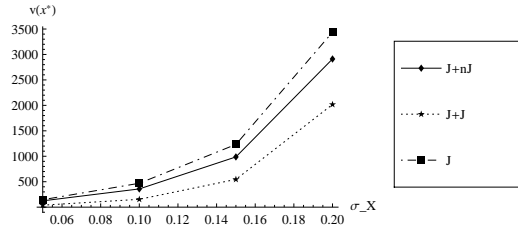


Figure 3: Behavior of the investment opportunity value as a function of the volatility of the demand volatility (σ_X)

In Figure 3 we can assess the impact of σ_X in the investment opportunity value, for 3 scenarios: J; J+nJ; J+J. In all the scenarios, higher uncertainty upon HSR demand results in higher investment opportunity values. Note that adding uncertainty upon investment expenditures reduces the sensibility of the investment opportunity value to changes in σ_X . In accordance with our intuition, these findings corroborate the smaller values for the investment opportunity and option to defer, when uncertainty upon investment expenditures with or without jumps is added to the valuation framework. For longer delays in investment decision, the uncertainty and possibility of unexpected shocks on investment expenditures will result in higher investment expenditures to implement the investment in the future, reducing the net cash-flows.

5 Conclusions and extensions

This paper developed a framework to determine the optimal timing to invest in HSR, facing uncertainty upon investment expenditures and demand, both influenced by unexpected conjecture shocks. Those extension were made, given the need to design an adequate framework for HSR investments in an environment of evolving technology and production improvements.

Compared to Couto et al. (2012), the inclusion of uncertainty upon investment expenditures led to a considerable anticipation of the trigger value. Our findings support intuition set up in previous downstream research, for instance McDonald and Siegel (1986) and Dixit and Pindyck (1994) for theoretical development in real options, and Bowe and Lee (2004), Pereira et al. (2006) and Pimentel et al. (2012) for using ROA in transportation investments.

The numerical illustration provides support to our findings and the simulation of some important input parameters demonstrates the consistency of the framework. Joint positive shocks in demand and in uncertainty upon increasing investment expenditures led to a decrease in the trigger value around 46%, in comparison with deterministic investment expenditures. Adding stochastic investment expenditures with no jumps is responsible for 49% decrease in the trigger value, meaning that the additional jumps upon investment expenditures is responsible for a slightly increase

in the trigger value.

Higher uncertainty means more relevance for ROA's application, as shown in this research. Our research provides an economic basis and financial rationale to the decision to invest in the HSR project, in an environment exposed to uncertainty technological progress and conjectural impacts.

Several extensions of the model in order to incorporate additional uncertainty factors and competitiveness effect may be conducted in the future, at expenses of additional complexity and the use of deeper numerical methods.

References

- Bjork, T. (2009). *Arbitrage Theory in Continuous Time*. Oxford: Finance.
- Bowe, M. and D. Lee (2004). Project evaluation in the presence of multiple embedded real options: Evidence from the taiwan high-speed rail project. *Journal of Asian Economics* 15, 71–98.
- Brandao, L. (2002). *Uma Aplicacao da Teoria das Opcoes Reais em Tempo Discreto para Avaliacao de uma Concessao Rodoviaria no Brasil*. Ph. D. thesis, PUC-Rio, Brasil.
- Couto, G., C. Nunes, and P. Pimentel (2012). High speed rail transport valuation and conjuncture shocks. *European Journal of Finance* DOI:10.1080/1351847X.2012.665377.
- Dixit, A. and R. Pindyck (1994). *Investment Under Uncertainty*. Princeton University Press.
- Huisman, K. (2000). *Technology Investment: a Game Theoretic Real Options*. Ph. D. thesis, Tilburg University, Department of Econometrics, CentER Dissertation Series Centre for Quantitative Methods in Eindhoven, Tilburg, The Netherlands.
- Kloeden, P. E. and E. Platen (1999). *Numerical Solution of Stochastic Differential Equations (Stochastic Modelling and Applied Probability)*. Springer.
- McDonald, R. and D. Siegel (1986). The value of waiting to invest. *The Quarterly Journal of Economics* 101(4), 707–728.
- Merton, R. (1976a). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125–144.
- Merton, R. (1976b). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125–144.
- Oksendal, B. (1998). *Stochastic Differential Equations: an Introduction with Applications*. Milan: Springer, 5th edition.
- Pereira, P., A. Rodrigues, and M. Armada (2006). The optimal timing for the construction of an international airport: A real options approach with multiple stochastic factors and shocks. 10th Real Options Conference, New York.

- Pimentel, P., J. Pereira, and G. Couto (2012). High speed-rail transport valuation. *The European Journal of Finance* 18(2), 167–183.
- Rose, S. (1998). Valuation of interacting real options in a tollroad infrastructure project. *The Quarterly Review of Economics and Finance* 38, 711–723.
- Ross, S. (1996). *Stochastic Processes*. New York: Wiley.
- Salahaldin, L. and T. Granger (2005). Investing in sustainable transport to relieve air pollution under population-growth uncertainty. 9th Real Options Conference, Paris.
- Smit, H. (2003). Infrastructure investment as real options game: The case of european airport expansion. *Journal of Transport Geography* 32, 27–57.
- Tourinho, O. (1979). *The Valuation of Reserves of Natural Resources: An Option Pricing Approach*. Ph. D. thesis, University of California, Berkeley.