THE ELLIPTICAL AND THE HYPERBOLICAL RANGE THEOREMS REVISITED

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Abstract. A geometrical proof of the Hyperbolical Range Theorem, concerning the numerical range of linear operators on 2-dimensional Krein spaces, is given. The classical Elliptical Range Theorem, which is the correspondent result for Hilbert spaces, is also obtained using the same technique. Both proofs depend on the geometrical properties of the plane algebraic curve that generates the numerical range as its convex hull or as its pseudo-convex hull. Keywords: numerical range, generalized numerical range, indefinite inner product space, plane algebraic curve. AMS subject classification: 15A60, 15A63, 46C20.

1 INTRODUCTION. In recent years, properties of the numerical range of an operator on an indefinite inner product space have been investigated by several authors (see Bebiano et al (1,2), Li and Rodman (8,9) and Li et al (10)). The concept has applications to many subjects, such as quantum systems and quantum computation. The investigation of the geometrical properties of the indefinite numerical range is essential for the development of this subject.

Let M_n be the algebra of $n \times n$ complex matrices and I_m the identity matrix of order m. The matrix $J = I_r \oplus -I_{n-r}$, $0 < r < n$, endows $\mathbb{C}^n$ with an indefinite inner product defined by $\langle x, y \rangle_J = y^* Jx$, $x, y \in \mathbb{C}^n$. The *J-numerical range* of $A \in M_n$ is denoted and defined by

$$W_J(A) = \left\{ \frac{\langle Ax, x \rangle_J}{\langle x, x \rangle_J} : x \in \mathbb{C}^n, \langle x, x \rangle_J \neq 0 \right\}. \tag{1.1}$$

If $J = \pm I_n$, then (1.1) reduces to the well known *classical numerical range* or *field of values* of A, simply denoted by $W(A)$.

We briefly recall some known properties. For $A \in M_n$, $W(A)$ is a compact set (Horn and Johnson (5)), while $W_J(A)$ may not be closed and is either unbounded or a singleton (10). For $\lambda \in \mathbb{C}$, $W_J(A) = \{\lambda\}$ if and only if $A = \lambda I_n$. The Toeplitz-Hausdorff Theorem states that the field of values of an operator is convex (5). On the other hand, $W_J(A)$ is the union of the convex sets

$$W_J(A) = W_J^+(A) \cup W_{-J}^+(A), \tag{1.2}$$

where $W_J^{\pm}(A) = \{\langle Ax, x \rangle_J : x \in \mathbb{C}^n, \langle x, x \rangle_J = \pm 1\}$, being $W_{-J}^+(A) = -W_J^-(A)$ (10). Moreover, $W_J(A)$ is *pseudo-convex* (10); that is, for any pair of distinct points $x, y \in W_J(A)$, if x, y belong to the same convex set in (1.2), $W_J^+(A)$ or $W_{-J}^+(A)$, then $W_J(A)$ contains the closed line segment joining x and y; otherwise, $W_J(A)$ contains the two closed half-lines of the line defined by x and y with endpoints x and y.

The set $W_J(A)$ is *J-unitarily invariant*, that is, $W_J(A) = W_J(U^{-1}AU)$, for any $U \in M_n$ satisfying $U^{-1} = JU^*J$. When $J = \pm I_n$, this property reduces to the unitary invariance of $W(A)$. Also, the following holds

$$W_J(\alpha I_n + \beta A) = \alpha + \beta W_J(A), \quad \alpha, \beta \in \mathbb{C}. \tag{1.3}$$

For $A \in M_2$, Murnaghan (11) proved that $W(A)$ is an elliptical disc (possibly degenerate) whose foci are the eigenvalues of A, α_1 and α_2. The major and minor axes are of length

$$\sqrt{\mathrm{tr}(A^*A) - 2\,\mathrm{Re}(\bar{\alpha}_1\alpha_2)} \quad \text{and} \quad \sqrt{\mathrm{tr}(A^*A) - |\alpha_1|^2 - |\alpha_2|^2},$$

respectively. This result is the well-known Elliptical Range Theorem. In the indefinite case, for $A \in M_2$ and $J = \mathrm{diag}(1,-1)$, the Hyperbolical Range Theorem (1) states that $W_J(A)$ is bounded by the hyperbola (possibly degenerate), with foci at α_1 and α_2, and transverse and non-transverse axes of length

$$\sqrt{\mathrm{tr}(JA^*JA) - 2\,\mathrm{Re}(\bar{\alpha}_1\alpha_2)} \quad \text{and} \quad \sqrt{|\alpha_1|^2 + |\alpha_2|^2 - \mathrm{tr}(JA^*JA)},$$

respectively. For the degenerate cases, $W_J(A)$ may be a singleton, a line, a subset of a line, the whole complex plane, or the complex plane except a line.

The 2×2 case is particular important, because it allows the derivation of many properties of the general $n\times n$ case. The description of $W_J(A)$, when $A \in M_n$ and $n > 2$, is not an easy task. Some authors have characterized (generalized) numerical ranges for some special classes of matrices. In certain cases, $W(A)$ and $W_J(A)$ still are an elliptical disc and a hyperbola and its interior, independently of the size of A.

Most proofs of the Elliptical and of the Hyperbolical Range Theorems are computational and quite involved (see Donoghue (4) and (5), (1)). The purpose of this note is to present a simple and unified proof of these results. Li (7) presented an elegant proof of the Elliptical Range Theorem, reducing the problem to the circular case. Our approach is based on the study of the associated curve, in the vein of Kippenhahn's investigation (6).

2 THE ASSOCIATED CURVE OF $W_J(A)$. Consider the *J-Cartesian decomposition* of $A \in M_n$, $A = H^J + iK^J$, where $H^J = (A + JA^*J)/2$ and $K^J = (A - JA^*J)/(2i)$ are *J-Hermitian* matrices, that is, $H^J = J(H^J)^*J$ and $K^J = J(K^J)^*J$. If $J = \pm I_n$, we obtain the well known *Cartesian decomposition* of A, being $H = H^J$ and $K = K^J$ Hermitian matrices. It can be easily seen that the projections of $W_J(A)$ on the lines passing through the origin and defining an angle θ with the real axis are given by $W_J(\cos\theta\, H^J + \sin\theta\, K^J)$. In particular, the projections of $W_J(A)$ on the real and imaginary axes are given by $W_J(H^J)$ and $W_J(K^J)$, respectively.

A *supporting line* of a convex set $K \subseteq \mathbb{C}$ is a line that intersects *K* at least at one point and that defines two half-planes, such that one of them does not contain any point of *K*. The supporting lines of $W_J(A)$ are the supporting lines of the convex sets $W_J^+(A)$ and $W_{-J}^+(A)$. As proved in (1) (cfr. Psarrakos (13)), if $ux + vy + w = 0$ is the equation of a supporting line of $W_J(A)$, then

$$F_A^J(u,v,w)=\det(uH^J+vK^J+wI_n)=0. \quad (2.1)$$

Since $F_A^J(u,v,w)$ is a homogeneous polynomial of degree *n*, (2.1) may be viewed as the line equation of an algebraic curve on the complex projective plane $P\mathbb{C}^2$. Considering the dual curve,

$$\Gamma^*=\{(u:v:w)\in P\mathbb{C}^2: F_A^J(u,v,w)=0\},$$

we may determine by dual arguments

$$\Gamma=\{(x:y:z)\in P\mathbb{C}^2: xu+yv+zw \text{ is a tangent of } \Gamma^*\},$$

whose real affine view

$$C_J(A)=\{(x,y)\in\mathbb{R}^2:(x:y:1)\in\Gamma\}$$

is called the *associated curve* of $W_J(A)$. If $J=\pm I_n$, $C_J(A)$ is simply denoted by $C(A)$ and is called the associated curve or *boundary generating curve* of $W(A)$.

It can be easily seen that the polynomial $F_A^J(u,v,w)$ has real coefficients, and that the real *foci* of the complex algebraic curve defined by the line equation (2.1) are the *n* eigenvalues of *A*. The complex curve contains $C_J(A)$ and has class *n*, that is, through a general point in the plane there are *n* lines (may be complex) tangent to that curve. (For details on plane algebraic curves, we refer to Brieskorn and Knörrer (3).)

In the classical case, the knowledge of $C(A)$ allows the complete characterization of $W(A)$. In the indefinite version, the same happens in the case of non-degeneracy of $W_J(A)$. For $J=\pm I_n$, Kippenhahn proved that $W(A)$ is the closed convex hull of $C(A)$ (6). For $J\neq\pm I_n$, if there exists $\theta\in[0,2\pi]$ such that the *n* eigenvalues $\lambda_1(\theta),\dots,\lambda_n(\theta)$ of $\cos\theta\, H^J+\sin\theta\, K^J$, where $A=H^J+iK^J\in M_n$, are real and have an associated basis of eigenvectors, $u_1(\theta),\dots,u_n(\theta)$ such that $\langle u_k(\theta),u_k(\theta)\rangle_J\neq 0$, $k=1,\dots,n$, then

$$x_k(\theta)=\frac{\langle Au_k(\theta),u_k(\theta)\rangle_J}{\langle u_k(\theta),u_k(\theta)\rangle_J}\in C_J(A),\quad k=1,\dots,n.$$

Considering all the different directions θ satisfying the above requisites, the set $W_J(A)$ is the pseudo-convex hull of the points so obtained (2), that is, for any two points, $x_r(\theta)$ and $x_s(\theta)$, take the closed line segment defined by them if $\langle u_r(\theta),u_r(\theta)\rangle_J\langle u_s(\theta),u_s(\theta)\rangle_J>0$; and take the two rays $\{\alpha x_r(\theta)+(1-\alpha)x_s(\theta):\alpha\leq 0 \text{ or } \alpha\geq 1\}$, if $\langle u_r(\theta),u_r(\theta)\rangle_J\langle u_s(\theta),u_s(\theta)\rangle_J<0$. The degenerate cases are obtained when such procedure is not possible.

For $A=H^J+iK^J\in M_2$, easy computations show that

$$F_A^J(u,v,w)=\det(H^J)u^2+\det(K^J)v^2+w^2+\operatorname{Re}\operatorname{tr}(A)uw+\operatorname{Im}\operatorname{tr}(A)vw+\operatorname{Im}\det(A)uv.$$

Applying (1.3), without loss of generality we may consider *A* with zero trace and real determinant. Therefore, we may take

$$F_A^J(u,v,w)=\det(H^J)u^2+\det(K^J)v^2+w^2. \quad (2.2)$$

Thus, the study of $C_J(A)$ depends only on the determinants of H^J and K^J.

Proposition 1. *Let C be the real part of the algebraic curve defined in homogeneous line coordinates by $G(u,v,w)=c_uu^2+c_vv^2+w^2$, for $c_u,c_v\in\mathbb{R}$.*

(a) If $c_u = c_v = 0$*, then* $C = \{0\}$*;*

(b) If $c_u < 0$ *and* $c_v = 0$*, then* $C = \{(-\sqrt{|c_u|},0),(\sqrt{|c_u|},0)\}$*;*

(c) If $c_u > 0$ *and* $c_v = 0$*, then* $C = \mathbb{R}$*;*

(d) If $c_u = 0$ *and* $c_v < 0$*, then* $C = \{(0,-\sqrt{|c_v|}),(0,\sqrt{|c_v|})\}$*;*

(e) If $c_u = 0$ *and* $c_v > 0$*, then* $C = i\mathbb{R}$*;*

(f) If $c_u > 0$ *and* $c_v > 0$*, then* $C = \varnothing$ *(an imaginary ellipse);*

(g) If $c_u < 0$ *and* $c_v < 0$*, then* C is the ellipse centred at the origin with major and minor axes on the coordinate axes, of lengths $2\sqrt{\max\{|c_u|,|c_v|\}}$ and $2\sqrt{\min\{|c_u|,|c_v|\}}$, respectively;

(h) If $c_u < 0$ *and* $c_v > 0$*, then* C is the hyperbola centred at the origin with transverse and non-transverse axes on the real and imaginary axes of length $2\sqrt{|c_u|}$ and $2\sqrt{|c_v|}$, respectively;

(i) If $c_u > 0$ *and* $c_v < 0$*, then* C is the hyperbola centred at the origin with transverse and non-transverse axes on the imaginary and real axes of length $2\sqrt{|c_v|}$ and $2\sqrt{|c_u|}$, respectively.

Proof. (a) If $c_u = c_v = 0$, then $G(u,v,w) = 0$ if and only if $w = 0$, and the solution of this equation is given by the pencil of lines $(u:v:0)$ passing through the point $(0:0:1)$. As usual, we identify this pencil with the point $(0:0:1)$, and so C is the origin of the affine plane.

(b) If $c_v = 0$, it follows that $G(u,v,w) = 0$ if and only if $w = \pm\sqrt{-c_u}\,u$. If $c_u < 0$, then we get the pencils of lines $(u:v:\sqrt{|c_u|}u)$ and $(u:v:-\sqrt{|c_u|}u)$, which pass through the points $(-\sqrt{|c_u|}:0:1)$ and $(\sqrt{|c_u|}:0:1)$, respectively. Identifying the pencils with the respective points of the real axis the result follows.

(c) If $c_v = 0$ and $c_u > 0$, then the unique real solution of the equation $G(u,v,w) = 0$ is obtained for $u = 0$ and $w = 0$. This solution is the line with coordinates $(0:1:0)$, i.e., the real axis.

The proofs of (d) and (e) are analogous to those of (b) and (c), respectively.

(f) If $c_u > 0$ and $c_v > 0$, then $G(u,v,w) = 0$ if and only if $|c_u|u^2 + |c_v|v^2 + w^2 = 0$, and this equation does not have real solutions. Since $(0:0:0)$ does not represent any line in the projective plane, C is the empty set.

(g) Suppose that $c_u < 0$ and $c_v < 0$. Thus, $G(u,v,w) = -|c_u|u^2 - |c_v|v^2 + w^2$. To obtain the equation of the algebraic curve in point coordinates, let $w = 1$ and $u = (-1 - vy)/x$. We easily get

$$\frac{\delta G}{\delta v}(x,y,v) = -2|c_u|\frac{y + y^2 v}{x^2} - 2|c_v|v.$$

Solving the equation $\delta G/\delta v(x,y,v) = 0$ with respect to v, we get

$$v = -\frac{|c_u|y}{|c_v|x^2 + |c_u|y^2} \quad \text{and} \quad u = -\frac{|c_v|x}{|c_v|x^2 + |c_u|y^2}.$$

Substituting the expressions of u and v in $G(u,v,1) = 0$, it follows that

$$\frac{x^2}{|c_u|} + \frac{y^2}{|c_v|} = 1.$$

By analogous arguments, (h) and (i) follow, and the Proposition is proved. □

3 PROOF OF THE ELLIPTICAL RANGE THEOREM.

Theorem 1. *Let* $A = H + iK \in M_2$, *such that* $\operatorname{tr}(A) = 0$ *and* $\operatorname{Im}\det(A) = 0$. *Consider* $h = \det(H)$ *and* $k = \det(K)$. *The following possibilities may occur:*

- *(a) If* $h = k = 0$, *then* $C(A) = \{0\}$ *and* $W(A) = \{0\}$;
- *(b) If* $h \neq 0, k = 0$, *then* $C(A) = \{(-\sqrt{|h|}, 0), (\sqrt{|h|}, 0)\}$ *and* $W(A) = \left[-\sqrt{|h|}, \sqrt{|h|}\right]$;
- *(c) If* $h = 0, k \neq 0$, *then* $C(A) = \{(0, -\sqrt{|k|}), (0, \sqrt{|k|})\}$ *and* $W(A) = i\left[-\sqrt{|k|}, \sqrt{|k|}\right]$;
- *(d) If* $h \neq 0, k \neq 0$, *then* $C(A)$ *is the ellipse centred at the origin with major and minor axes on the coordinate axes of length* $2\sqrt{\max\{|h|, |k|\}}$ and $2\sqrt{\min\{|h|, |k|\}}$, *respectively, and* $W(A)$ *is the ellipse and its interior.*

Proof. Since H and K are Hermitian and $\operatorname{tr}(H) = \operatorname{tr}(K) = 0$, the eigenvalues of H and K are real and symmetric. Therefore, $\det(H) \leq 0$ and $\det(K) \leq 0$. Recalling that $W(A)$ is the closed convex hull of $C(A)$, the Theorem follows from Proposition 1 (a), (b), (d) and (g). □

4 PROOF OF THE HYPERBOLICAL RANGE THEOREM.

For a *J*-Hermitian matrix $A \in M_n$, $J \neq \pm I_n$, it is well known that

$$W_J^+(A) = \{x \in \mathbb{R} : t(x+i) \in W(JA + iJ), \text{ for some } 0 < t \leq 1\} \tag{4.1}$$

and

$$W_{-J}^+(A) = \{x \in \mathbb{R} : t(-x-i) \in W(JA + iJ), \text{ for some } 0 < t \leq 1\}. \tag{4.2}$$

From (4.1) and (4.2), it follows that $W_J^+(A)$ is a right half-line $[m_1, +\infty[$, or $]m_1, +\infty[$, for certain $m_1 \in \mathbb{R}$, if and only if $W_{-J}^+(A)$ is a left half-line $]-\infty, m_2]$, or $]-\infty, m_2[$, for some $m_2 \in \mathbb{R}$. The endpoints of these half-lines are eigenvalues of *A*. On the other hand, $W_J^+(A) = \mathbb{R}$ if and only if $W_{-J}^+(A) = \mathbb{R}$. (For more details, we refer to Bebiano et al (12).)

Lemma 1. *Let* $J = \operatorname{diag}(1, -1)$ *and* $0 \neq A = (a_{ij}) \in M_2$ *a J-Hermitian matrix such that* $\operatorname{tr}(A) = 0$. *The following holds:*

(a) If $\det(A) < 0$, *then*

$$C_J(A) = \{(-\sqrt{|\det(A)|}, 0), (\sqrt{|\det(A)|}, 0)\} \quad \text{and} \quad W_J(A) = \mathbb{R} \setminus \left]-\sqrt{|\det(A)|}, \sqrt{|\det(A)|}\right[;$$

(b) If $\det(A) = 0$, *then* $C_J(A) = \{0\}$ *and* $W_J(A) = \mathbb{R} \setminus \{0\}$;

(c) If $\det(A) > 0$, *then* $C_J(A) = \mathbb{R}$ *and* $W_J(A) = \mathbb{R}$.

Proof. Recalling (2.2), $C_J(A)$ is given in homogeneous line coordinates by the equation $\det(A)u^2 + w^2 = 0$, and is characterized by Proposition 1 (a)–(c).
Without loss of generality, we may consider

$$A = \begin{pmatrix} c & d \\ -d & -c \end{pmatrix},$$

being $c \in \mathbb{R}$ and $d \geq 0$. To achieve this conclusion, we use the diagonal matrix $U = \mathrm{diag}(e^{i\alpha}, e^{-i\alpha})$, $\alpha = (\arg a_{12})/2$, and the *J*-unitary invariance of $W_J(A)$. To characterize $W(JA + iJ)$, we consider the associated curve $C(JA + iJ)$, defined in homogeneous line coordinates by the equation $\det(uJA + vJ + wI_2) = 0$. By easy calculations, we find the point equation of $C(JA + iJ)$:

$$(x - c)^2 + d^2 y^2 = d^2. \tag{4.3}$$

If $d \neq 0$, we obtain an ellipse with centre $(c, 0)$ and semi-axes of length d and 1. Additionally, if $\det(A) > 0$, then $d > |c|$, and the origin is an interior point of $W(JA + iJ)$. By (4.1) and (4.2), it follows that $W_J(A) = W_J^+(A) = W_{-J}^+(A) = \mathbb{R}$. If $\det(A) = 0$, then $d = |c|$, and the origin is a point of the ellipse. The imaginary axis is the tangent to the ellipse at the origin, and it is a supporting line of $W(JA + iJ)$. By (4.1) and (4.2), $W_J^+(A)$ and $W_{-J}^+(A)$ are the two open half-lines of the real line with endpoint 0, and $W_J(A) = \mathbb{R} \setminus \{0\}$. If $\det(A) < 0$, then $d < |c|$, and $0 \notin W(JA + iJ)$. From (4.3), we deduce the equations of the tangents to the ellipse that pass through the origin: $x = \pm\sqrt{|\det(A)|}\, y$. Then, $W_J^+(A)$ and $W_{-J}^+(A)$ correspond to the two closed half-lines of the real line with endpoints $\pm\sqrt{|\det(A)|}$, and

$$W_J(A) = \mathbb{R} \setminus]-\sqrt{|\det(A)|}, \sqrt{|\det(A)|}[. \tag{4.4}$$

Finally, if $d = 0$ in (4.3), then $W(JA + iJ)$ is the line segment connecting the points $(c, \pm 1)$. Since $A \neq 0$, it follows that $c \neq 0$. Then, $0 \notin W(JA + iJ)$, and $x = \pm cy$ are the supporting lines that pass through the origin. Since $\det(A) = -c^2 < 0$, it is obvious that (4.4) still holds. □

In contrast to the classical case, if $J = \mathrm{diag}(1, -1)$, the spectrum of a *J*-Hermitian matrix may be non-real. Therefore, if $A = H^J + iK^J$ is the *J*-Cartesian decomposition of a matrix A with real determinant and zero trace, it is possible that $\det(H^J) > 0$ or $\det(K^J) > 0$. However, other possibilities for $\det(H^J)$ and $\det(K^J)$ can not occur simultaneously.

Lemma 2. *Let* $J = \mathrm{diag}(1, -1)$ *and* $A = H^J + iK^J \in M_2$. *Suppose that* $\mathrm{tr}(A) = 0$ *and* $\mathrm{Im}\det(A) = 0$.

(a) The inequalities $\det(H^J) < 0$ *and* $\det(K^J) < 0$ *can not occur simultaneously;*

(b) If $\det(H^J) < 0$ *and* $\det(K^J) = 0$, *then* $K^J = 0$;

(c) If $\det(H^J) = 0$ *and* $\det(K^J) < 0$, *then* $H^J = 0$.

Proof. (a) Let $A = (a_{ij})$. Since $\mathrm{tr}(A) = 0$, it follows that $a_{22} = -a_{11}$. From $\det(H^J) < 0$ and $\det(K^J) < 0$, we easily get the inequalities

$$\frac{1}{4}(a_{12} \mp \bar{a}_{21})(\bar{a}_{12} \mp a_{21}) < \pm\frac{1}{4}(a_{11} \pm \bar{a}_{11})^2.$$

The left and the right hand sides of the above inequalities are non-negative, and by multiplication we find

$$\frac{1}{16}(a_{12}{}^2 - \bar{a}_{21}{}^2)(\bar{a}_{12}{}^2 - a_{21}{}^2) < -\frac{1}{16}(a_{11}{}^2 - \bar{a}_{11}{}^2)^2. \tag{4.5}$$

Since $\det(A)$ is a real number, $-a_{11}{}^2 - a_{12}a_{21} = -\bar{a}_{11}{}^2 - \bar{a}_{12}\bar{a}_{21}$, and so

$$({a_{12}}^2-{\bar{a}_{21}}^2)({\bar{a}_{12}}^2-{a_{21}}^2)<-(a_{12}a_{21}-\bar{a}_{12}\bar{a}_{21})^2.$$

Then $|a_{12}|^4+|a_{21}|^4-2|a_{12}|^2|a_{21}|^2<0$, a contradiction.
(b) If $a_{11}\neq\bar{a}_{11}$, then (4.5) still holds, and we obtain again a contradiction. Therefore, a_{11} must be a real number. From $\det(K^J)=0$, it follows that $a_{12}=-\bar{a}_{21}$, and $K^J=0$.
(c) The proof is analogous to (b). □

Theorem 2. *For $A=(a_{ij})$ under the assumptions of Lemma 2, with $h=\det(H^J)$ and $k=\det(K^J)$, the following holds:*

- *(a) If $h=k=0$, then $C_J(A)=\{0\}$. If $A=0$, then $W_J(A)=\{0\}$; otherwise, $W_J(A)$ is the line defined by a_{11} and $-a_{11}$, except the origin;*
- *(b) If $h<0,k=0$, then $C_J(A)=\{(-\sqrt{|h|},0),(\sqrt{|h|},0)\}$ and $W_J(A)=\mathbb{R}\setminus\,]-\sqrt{|h|},\sqrt{|h|}[$;*
- *(c) If $h>0,k=0$, then $C_J(A)=\mathbb{R}$. If $K^J=0$, then $W_J(A)=\mathbb{R}$; otherwise, $W_J(A)=\mathbb{C}\setminus\mathbb{R}$;*
- *(d) If $h=0,k<0$, then $C_J(A)=\{(0,-\sqrt{|k|}),(0,\sqrt{|k|})\}$ and $W_J(A)=i\,\mathbb{R}\setminus\,]-\sqrt{|k|},\sqrt{|k|}[$;*
- *(e) If $h=0,k>0$ then $C_J(A)=i\mathbb{R}$. If $H^J=0$, then $W_J(A)=i\mathbb{R}$; otherwise, $W_J(A)=\mathbb{C}\setminus i\mathbb{R}$;*
- *(f) If $h>0,k>0$, then $C_J(A)=\varnothing$ and $W_J(A)=\mathbb{C}$;*
- *(g) If $h<0,k>0$, then $C_J(A)$ is the hyperbola centred at the origin with transverse and non-transverse axes on the real and imaginary axes of length $2\sqrt{|h|}$* and *$2\sqrt{|k|}$, respectively; $W_J(A)$ is the hyperbola and its interior;*
- *(h) If $h>0,k<0$, then $C_J(A)$ is the hyperbola centred at the origin with transverse and non-transverse axes on the imaginary and real axes of length $2\sqrt{|k|}$* and *$2\sqrt{|h|}$, respectively; $W_J(A)$ is the hyperbola and its interior.*

Proof. By Proposition 1 and Lemma 2(a), the characterization of $C_J(A)$ is clear.
(a) Since $h=k=0$, $C_J(A)=\{0\}$. Suppose that $K^J=0$. If $H^J=0$, then $A=0$ and $W_J(A)=\{0\}$. Otherwise, $A=H^J$, $a_{11}\neq 0$ is real and Lemma 1 implies that $W_J(A)=\mathbb{R}\setminus\{0\}$. Suppose now that $K^J\neq 0$. Let $B=e^{-i\theta}A\neq 0$, where $\theta=\arg a_{11}$. From $h=k=0$, we obtain $(\operatorname{Re}a_{11})^2=1/4|a_{12}-\bar{a}_{21}|^2$ and $(\operatorname{Im}a_{11})^2=1/4|a_{12}+\bar{a}_{21}|^2$. It follows that

$$\operatorname{Re}({a_{11}}^2)=\operatorname{Re}(a_{11})^2-\operatorname{Im}(a_{11})^2=-\operatorname{Re}(a_{12}a_{21}). \tag{4.6}$$

Since $\det(A)=-{a_{11}}^2-a_{12}a_{21}\in\mathbb{R}$, (4.6) implies that $\det(A)=0$. It is also easily seen that $|a_{11}|=|a_{12}|=|a_{21}|$. Therefore, B is *J*-Hermitian and $\det(B)=0$. By Lemma 1, $W_J(B)=\mathbb{R}\setminus\{0\}$, and $W_J(A)=e^{i\theta}\mathbb{R}\setminus\{0\}$.
(b) Suppose that $h<0$ *and* $k=0$. By Lemma 2(b), $K^J=0$. Thus, $A=H^J$ and the result follows from Proposition 1 and Lemma 1.
(c) Consider $h>0$ *and* $k=0$. Then $C_J(A)$ is the real axis. If $K^J=0$, then $A=H^J$ and by Lemma 1, $W_J(A)=\mathbb{R}$. If $K^J\neq 0$, by Lemma 1, $W_J(K^J)=\mathbb{R}\setminus\{0\}$. Then, the projection of $W_J(A)$ on the imaginary axis is the imaginary axis with the origin deleted. Thus, $W_J(A)$ is the whole complex plane with the real axis deleted.
The proofs of (d) and (e) are analogous to (b) and (c), respectively.
(f) Suppose that $h>0$ *and* $k>0$. Lemma 1 ensures that $W_J(H^J)=\mathbb{R}$ and $W_J(K^J)=\mathbb{R}$. By

Proposition 1, $C_J(A)=\varnothing$, and so $W_J(A)=\mathbb{C}$.

(g) Let $h<0$ *and* $k>0$. By Lemma 1, $W_J(H^J)=\mathbb{R}\setminus\left]-\sqrt{|h|},\sqrt{|h|}\right[$ and $W_J(K^J)=\mathbb{R}$. By Proposition 1, $C_J(A)$ is the hyperbola centred at the origin with transverse axis on the real axis of length $2\sqrt{|h|}$,and non-transverse axis on the imaginary axis of length $2\sqrt{|k|}$. Then, $W_J(A)$ is the hyperbola and its interior.

(h) The proof is analogous to (g). □

In the theory of numerical ranges of Krein space operators, many open problems remain, such as the classification of $C_J(A)$ in the 3×3 case. For a general n, not much is known.

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