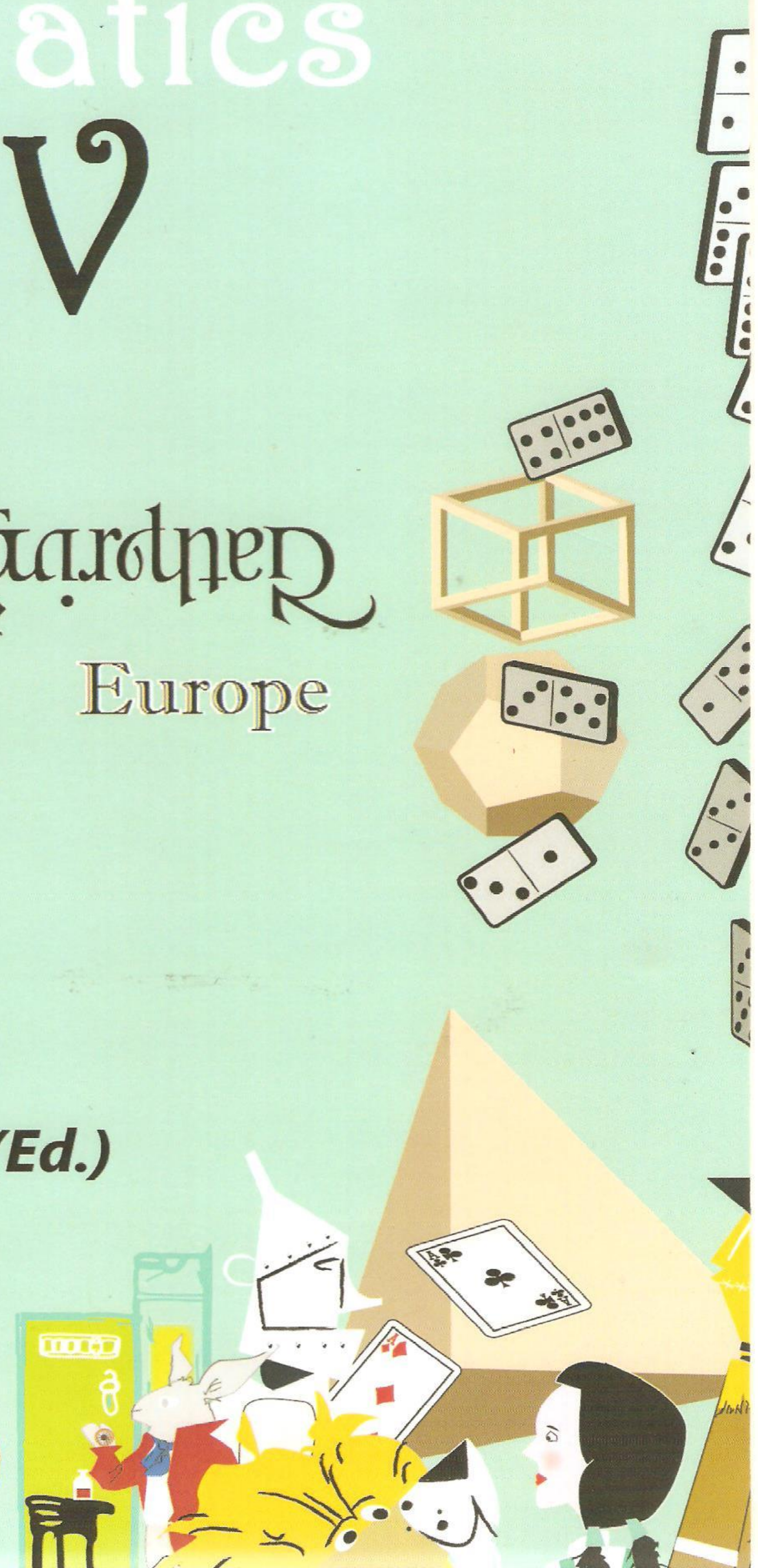
Recreational Mathematics Colloquium IV

Gathering for Gardner

Europe



Proceedings

Jorge Nuno Silva (Ed.)

PROCEEDINGS OF THE

RECREATIONAL MATHEMATICS COLLOQUIUM IV
Gathering for Gardner EUROPE

Jorge Nuno Silva (Ed.)

Associação Ludus

Lisboa, 2016

CONTENTS

MATHEMATICS AND FIBER ARTS

Maria do Carmo Martins
Department of Mathematics
University of Açores, Portugal
mika@uac.pt

Helena Sousa Melo
Department of Mathematics
University of Açores, Portugal
hmelo@uac.pt

Sandra Vinagre
Department of Mathematics
University of Évora, CIMA, Portugal
smv@uevora.pt

Abstract

Mathematics can be found all over the world, even in what could be considered an unrelated area, like fiber arts. In knitting, crochet, and counted-thread embroidery, we can find concepts of algebra, graph theory, number theory, geometry of transformations, and symmetry, as well as computer science. For example, many fiber art pieces embody notions related with groups of symmetry. In this work, we focus on two areas of Mathematics associated with knitting, crochet, and cross-stitch works – number theory and geometry of transformations.

Introduction

The relationship between the beauty of fiber arts and the formal and rigorous character of Mathematics seems strange. Surprisingly, the benefits of this connection is amazing and has been explored over time. In knitting, crochet, and cross-stitch, we study and apply concepts of number theory and transformations geometry, namely symmetry.

The mathematics behind works with cross-stitch, crochet, and knitting includes the transformation geometry, using the isometry and symmetry, and modular arithmetic, necessary to count points for the motif and its reproduction.

Usually mathematicians and crafters are trained to notice fine-grained distinction, namely, mathematicians deal with numbers, geometric shapes and abstracts relations, and crafters are prepared to create new pieces. However, both mathematicians and crafters, look towards harmony.

Our main challenge is in use of knitting, crochet, and cross-stitch to show mathematics and, at the same time, using the knowledge of mathematics to create new fiber art pieces. Many fiber art pieces contain ornamental patterns. The ornamental patterns on the plane are the rosette patterns, the frieze patterns, and the wallpaper patterns.

In this paper, we start by reviewing concepts of symmetry group on the plane, in particular, rosette patterns, frieze patterns, and wallpaper patterns. We finish this paper with conclusions and future work.

Symmetry groups in knitting, crochet and cross-stitch

A figure is a nonempty set of points. Let s and t be figures. Then s is congruent to t when overlapping coincide point by point. An isometry is a transformation in which the original figure is congruent to its image and that preserves distances. An isometry is invariant with respect to distance and, consequently, invariant with respect to angles. On the plane, reflections, rotations, translations or any combination of two, or more, of them are isometries. The dilation transformation is not an isometry because, in general, it does not preserve distances.

A symmetry is an isometry, that leaves figures invariant, and therefore is often a pattern formed by the repetition of some motif over and over again. The simplest types of symmetries are reflection or mirror symmetry, rotational symmetry and point symmetry associated with a hulfturn. The craftwork and art of most cultures involves symmetry, often in the design of backgrounds and border, see [2]. According to the symmetry groups, rosette group, frieze group and wallpaper group, we can classify a figure as a rosette, a frieze or a wallpaper, respectively. Normally, the notation of these patterns is the same of their ornamental group. In this work we follow the notation of László Fejes Tóth (1915-2005), see [1].

Rosette patterns

Rosette patterns are plane figures that have no translations and, at least, one point of the pattern is not moved by any of the symmetry transformation, which is called the rosette center. The rosette groups are finite groups of isommetries that can be divided into cyclic and dihedral groups. The former, represented by C_n, comprises n rotations equally spaced around the center; and the latter, denoted by D_n, contains the cyclic group (C_n) and includes n reflections through the center of rotation, where n is exactly the number of rotations symmetry or reflection symmetry. The center of rotation is also called n-center and it is the center of symmetry. Notice that $n = 360°/a$, where a is the minimum angle of rotation.

In rosette patterns, for knitting and crochet, it is possible to get the two types with several rotation angles around its center point, but this is not possible for cross-stitch. The main reason is the evenwave used to embroider the cross-stitch imposes the classification. So, in cross-stitch, we can only get six types, three cyclic groups and three dihedral groups, C_1, C_2, C_4, D_1, D_2 and D_4, because the magnitude of the smallest possible angle of the nonidentity rotation is 90°, see [2].

We remark that when a plane figure presents only the identity transformation, i.e., have neither reflections nor nonidentity translation, nor nonidentity rotation, this plane figure is called asymmetric and its classification is C_1. The identity is a rotation through angle 0°.

Figure 1a presents a dihedral group with exactly eight symmetries, four reflections and four rotations. The intersection of all four lines of symmetry is the center of the rotation and the

minimum angle between the lines of symmetry is 45°. The rotations are the identity (rotation angle 0°), rotation of 90°, halfturn (rotation of 180°), and rotation of 270°. According to this description the classification is D_4. We observe that in a dihedral group the number of reflections and rotations is the same.

Figure 1b depicts a cyclic group with six rotations, i.e., with a 6-center. The rotation angles are 0°, 60°, 120°, 180°, 240°, and 300°. Notice that there are no reflections. The smallest angle of the nonidentity rotation is 60°, so the classification is C_6, because $6 = 360°/60°$.

Finally, in Figure 1c, we observe that there exists just one line of symmetry, and consequently, there is only one rotation of 0°, because the number of reflections and rotations must be the same. Its classification is D_1.

(a) (b) (c)

Figure 1: Rosette patterns.

Frieze patterns

Around the frieze of an older building there is often a pattern formed by the repetition of some figure or motif over and over again in one direction.

Frieze patterns, frequently used for fabric and paper borders, comprise translations in one direction along one line considered the horizontal direction. A frieze group is group of isometries that fix a given line and whose translations form an infinite cyclic group. The essential property of an ornamental frieze pattern is that it is left fixed by some "smallest translation". There are seven types of frieze groups that involve translations in one direction; reflection with parallel or perpendicular axis to the vector of the translation; rotations of 180° (the halfturns) and the combination of them, i.e., glide reflections. We have all seven types of frieze patterns in any one of three fiber arts.

Figure 2a presents a frieze pattern with translational symmetry in one direction, halfturn symmetry and glide reflection symmetry. In addition, it has vertical line symmetry that can be obtained by composing a glide reflection with a hulfturn. These hulfturns determine all points of symmetries, i.e., the 2-centers. The lenght between two consecutive 2-centers is half the smallest lenght of the vector of the translation. Therefore, the frieze pattern is classified by F_2^2. If 2-centers are on the intersection of the vertical and horizontal axis of symmetry the frieze pattern is classified by F_2^1.

In Figure 2b we only have translation and glide reflection, that is no reflection. Its classification is F_1^3. The vector of this glide reflection is half of the translation vector. If the vector of the glide reflection and the vector of the translation are the same, then we have a reflection in the same direction of the translation, and its classification would be F_1^1, because all reflections symmetries are glide reflections symmetries when the vectors are the same.

In case of Figure 2c, we only have translational symmetry and halfturn symmetry. So, its classification is F_2.

We describe five of the seven types of frieze patterns. The cases missing have the classifications F_1 and F_1^2. In frieze pattern F_1 exists only translational symmetry and no other symmetry. In F_1^2 exists translational symmetry and vertical line symmetry.

(a) (b) (c)

Figure 2: Frieze patterns.

Wallpaper patterns

The knowledge of all possible wallpaper patterns goes back to the ancient times. A wallpaper pattern is a plane figure with two independent translational symmetries. In addtion to translational symmetry, wallpaper patterns can have line symmetry, glide reflection symmetry, and 180°, 120°, 90° or 60° rotational symmetry. Wallpaper groups are groups of symmetries of wallpaper patterns. A wallpaper group contains a sub-group of translations generated by two independent translations. There are sub-groups without rotations and sub-groups which rotation angle are 180°, 120°, 90° and 60°, corresponding to 2-center, 3-center, 4-center and 6-center, respectively. There are only seventeen different types of wallpaper patterns. There are four symmetry types of wallpaper patterns with no n-center, five symmetry types with 2-centers, three symmetry types with smallest rotation angle of 120° (3-center), three symmetry types with smallest rotation angle of 90° (4-center) and two symmetry types with smallest rotation angle of 60°, (6-center). Observe that the only rotational symmetries in a frieze group are halfturns. In the wallpaper group we have four rotational symmetry. The symmetries of a wallpaper pattern fix the set of n-centers.

Notice that a wallpaper group that has rotation angle with 4-center has no 3-center or 6-center, but has rotations of 2-center, because the rotation angle of 4-center is 90° and the rotation angle of 2-center is 180°. Similarly, a wallpaper group that has rotation angle of 6-center has no 4-center, but has rotation angle of 3-center and 2-center, because the rotation angle of 6-center is 60°, the rotation angle of 3-center is 120°, and the rotation angle of 2-center is 180°.

In wallpaper patterns there is a restriction to cross-stitch, because exist twelve of the seventeen. In fact, only four without rotations, five with rotations of 180°, and three with rotations of 90° are possible. The wallpaper patterns with rotation of 120° and 60° are impossible, since the evenwave

used to embroider the cross-stitch does not allow.

For tricot and crochet this restriction does not exist, since we can make all the seventeen patterns either continuously or with modules.

Figure 3 depicts all patterns with translational symmetries in two distinct directions and that, according to the rotation angle (which preserves the wallpaper patterns), receives a different type between the seventeen classifications.

In Figures 3a and 3b there are lines of symmetry and halfturn symmetries. However, there is a difference between them: in case of 3a all the centers around halfturn are on the axes of reflection, what does not occur in the case 3b. The classification is W_2^2 and W_2^1, for 3a and 3b, respectively.

In the case of 3c exists rotations of 90°, halfturns, and reflections. We observe that 2-centers are the middle point between two adjacent 4-centers. There are also lines of symmetry contained centers of rotation with the minimum angle between two lines of symmetry is 45°. The classification for the case 3c being W_4^1.

The classification of wallpaper patterns can be obtained throught a flowchart for recognition, see [1].

(a) (b) (c)

Figure 3: Wallpaper patterns.

Conclusions and future work

We have endeavored to maintain a level of accessibility that will allow anyone to understand what we did. We do encourage to stretch yourself by trying a new craft and learning some unknown mathematics. As we experiment new creations, we will appreciate the connections between the doing and learning of mathematics as an through craft.

This work is an approach to several possible lines of research and applications of mathematics to the fiber arts. There is a huge universe to observe and decode. The relationship between mathematics and art is unbounded. Given the academic, the enjoy and artistic skills of the authors, there are several topics still to be explored, having been made part of the connection with the transformation geometry, i.e., symmetry.

We hope to develop new approaches in this line of research, discovering and exploring curiosities not only in knitting, crochet, and cross-stitch but also in other areas of fiber art.

Acknowledgements

This work has been partially supported by the research center CIMA, FCT-Portugal funding program.

References

[1] George E. Martin, *Transformation Geometry: An Introduction to Symmetry*, Springer-Verlag New York Inc, 1982.

[2] Sarah-Marie Belcastro, Carolyn Yackel (ed.), *Making mathematics with needlework: ten papers and ten projects*, A. K. Peters, Wellesley, MA, 2008.